EXC 3

Every permutation of $[d]$ is obtained from a permutation of $[d-1]$ by introducing d into one of the d possible positions: at the beginning, at the end or in any of the $d-2$ spaces between the $d-1$ elements of the permutation of $[d-1]$. Since the permutations obtained in this way are $d!$, they are exactly the permutations of $[d]$ (no repetition).

Thus, we can count the number E(d,k) of permutations of $[d]$ with k descents by analyzing their relation to the permutations of $[d-1]$:
Let $(n_1, n_2, ..., n_{d-1})$ be a permutations of $[d-1]$ with t descents then there are 4 possible types of locations to introduce d: if we insert d at the beginning, we get $(d, n_1, n_2, ..., n_{d-1})$ a permutation of $[d]$ with $t+1$ descents if we insert d at the end, we get $(n_1, n_2, ..., n_{d-1}, d)$ a permutation of $[d]$ with t descents if we insert d in the middle of a descent (n_i, n_j), we get $(d, n_1, n_2, ..., n_i, d, n_j .., n_{d-1})$ a permutation of $[d]$ with t descents (the descent (n_i, n_j) converted into a non-descent (n_i, d) and a descent (d, n_j)) if we insert d in the middle of a non-descent (n_i, n_j), we get $(d, n_1, n_2, ..., n_i, d, n_j .., n_{d-1})$ a permutation of $[d]$ with $t+1$ descents (the non-descent (n_i, n_j) converted into a non descent (n_i, d) and a descent (d, n_j))

consequently, the permutations of $[d]$ with $k-1$ descends, are obtained by: introducing d into one of the $E(d-1, k-1)$ permutations of $[d-1]$ with $k-2$ descends in such a way that it rises the number of descents or by introducing d into one of the $E(d-1, k)$ permutations of $[d-1]$ with $k-1$ descends in such a way that it conserves the number of descents.

There are $n-k+1$ ways of rising the number of descends of a permutation of $[d-1]$ with $k-2$ descends (introducing d at the beginning or in the middle of one of the $(n-2)-(k-2)$ non-descents) There are k ways of conserving the number of descents of a permutation of $[d-1]$ with $k-1$ descends (introducing d at the end or in the middle of one of the $k-1$ descents.

since the number of descends either rises or stays the same when obtaining a permutation of $[d]$ from a permutation of $[d-1]$ by introducing d, these are the only ways of obtaining a permutation of $[d]$ with $k-1$ descends.

Thus, $E(d,k) = (n-k+1)E(d-1,k-1) + kE(d-1,k)$

Since $E(1,1) = 1 = A(11)$ and since $E(d,k)$ and $A(d,k)$ are defined by the same recursion, we have $E(d,k) = A(d,k)$ for all integers $1 \leq k \leq d$. (this can be proofed easily by induction on d, the base case is done, and $E(d,k), A(d,k)$ only depend on d,k and the previous cases with a smaller d)

To see $A(d,k) = A(d,d+1-k)$ for all integers $1 \leq k \leq d$, just notice that $A(d,d+1-k)$ represents the number of permutations of $[d]$ with $d-k$ descents. Since there is either a descent or an ascend between to numbers of a permutation, the permutations with $d-k$ descents are exactly those with$k-1$ ascends. Since there is a bijection ("taking it in reversed order") between the permutations with t ascends and the permutations with t descends, the number of the permutations with $k-1$ ascends is exactly the number of permutations with $k-1$ descends, that is $A(d,k)$.