2. Show that,

$$\sum_{t\geq 0}^{\infty}(t+1)^d z^t = \frac{A(d,1)z^0 + A(d,2)z^1 + \cdots A(d,d)z^{d-1}}{(1-z)^{d+1}}$$

where $A(d,k) = (d-k+1)A(d-1,k-1) + kA(d-1,k)$ and $A(d,k) = 0$ if $k \leq 0$ or $k \geq d+1$ for all $1 \leq k \leq d$

Proof. When d=0 we get that $\sum_{t\geq 0}^{\infty} z^t = \frac{1}{(1-z)}$. When d=1 we have that

$$\sum_{t\geq 0}^{\infty}(t+1)^1 z^t = \sum_{t\geq 0}^{\infty}\frac{d}{dz}z^{t+1} = \frac{d}{dz}\sum_{t\geq 0}^{\infty} z^{t+1} = \frac{d}{dz}(\frac{1}{1-z} - 1) = \frac{1}{(1-z)^2}$$

notice that $A(2,1)z^0 = (2A(1,0) + A(1,1))z^0 = (0 + A(1,1))z^0 = 1z^0 = 1$ which satisfiies the equality above. We shall proceed by induction and assume true for the d-1 case and now we need to show that equality holds for d.

$$\sum_{t\geq 0}^{\infty}(t+1)^d z^t = \sum_{t\geq 0}^{\infty}(t+1)^{d-1}\frac{d}{dz}z^{t+1} = \frac{d}{dz}\sum_{t\geq 0}^{\infty}(t+1)^{d-1}z^{t+1}$$

$$\frac{d}{dz}\left(z\left(\sum_{t\geq 0}^{\infty}(t+1)^{d-1}z^t\right)\right)$$

by our induction hypothosis we get,

$$\frac{d}{dz}\left(\frac{A(d-1,1)z^1+A(d-1,2)z^2+\cdots A(d-1,d-1)z^d}{(1-z)^{d+1}}\right)=\frac{d}{dz}\left(\frac{A(d-1,1)z^1}{(1-z)^{d+1}}+\frac{A(d-1,2)z^2}{(1-z)^{d+1}}+\cdots+\frac{A(d-1,d-1)}{(1-z)^{d+1}}\right.$$

where the derivative of each term is of the form,

$$\frac{d}{dz}\left(\frac{A(d-1,i)z^i}{(1-z)^{d+1}}\right)=\frac{iA(d-1,i)z^{i-1}+(d-i)A(d-1,i)z^i}{(1-z)^d}$$

for $1\leq i\leq d-1$. so the sum is of the form

$$\frac{A(d-1,i)z^0+\sum_{i=2}^{d-2}(A(d-1,i-1)+(d-i)A(d-1,i))z^{i-1}+((d-1)A(d-1,d-1))z^{d-1}}{(1-z)^d}$$

Notice that for $l=k-1$ we have that the coefficient of each z^l for $1\leq k\leq d$ is exactly of the form

$$A(d,k)z^{k-1}=(d-k+1)A(d-1,k-1)+kA(d-1,k)$$

note that when $k=1,l=0$ we have $A(d,1)=dA(d-1,0)+A(d-1,1)=0+A(d-1,1)$ and for $k=d,l=d-1$ we have $A(d,d)=A(d-1,d-1)+dA(d-1,d)=A(d-1,d-1)+0$. There we have the exact equality for d and so we are done. For this problem I worked with Ashley. $\square$