1. (Triangulations of $\Delta_{n-1} \times \Delta_{1}$ ) Describe all the triangulations of the product of simplices $\Delta_{n-1} \times \Delta_{1}$. Are they all regular?
Construction by induction:
Base case: Let $\mathrm{n}=2$
There are two ways to triangulate a rectangle, see below:


Inductive step: suppose we can triangulate $\Delta_{d-2} \times \Delta_{1}$, and let $\mathrm{n}=\mathrm{d}-1$ We can think of $\Delta_{d-1} \times \Delta_{1}$ as a prism with a simplex of dimension d-2 at either end. Let us then label the vertices of $\Delta_{d-1} \times \Delta_{1}$ as $\left\{p_{1}, \ldots, p_{d-1}, p_{1}^{\prime}, p_{d-1}^{\prime}\right\}$ where $\left\{p_{1}, \ldots, p_{d-1}\right\}$ and $\left\{p_{1}^{\prime}, \ldots, p_{d-1}^{\prime}\right\}$ are (d-2)-dimensional simplices.
Any triangulation must have some (d-1)-dimensional simplex with $\left\{p_{1}, \ldots, p_{d-1}\right\}$ as a facet. This simplex must contain exactly one of $\left\{p_{1}^{\prime}, \ldots, p_{d-1}^{\prime}\right\}$ as a vertex. There are precisely d-1 ways to choose this simplex.
Now consider the rest of the polytope with this simplex removed. This is simply a pyramid $\operatorname{pyr}\left(\Delta_{d-2} \times \Delta_{1}\right)$. Every triangulation of $\Delta_{d-2} \times \Delta_{1}$ corresponds bijectively with a triangulation of $\operatorname{pyr}\left(\Delta_{d-2} \times \Delta_{1}\right)$, thus we conclude that there are

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(d-1)\left(\text { number of ways to triangulate } \Delta_{d-2} \times \Delta_{1}\right)
$$

ways to triangulate $\Delta_{d-1} \times \Delta_{1}$.

Working from our base case we see that there are (n-1)! possible triangulations.

Now we can prove they are regular.
We know that for all polytopes, there exists at least one regular triangulation. Therefore by this theorem, at least one of our $(n-1)$ ! triangulations of $\Delta_{n-1} \times \Delta_{1}$ is regular.

Any triangulation isomorphic to this one is also regular, by symmetry, and the group $S_{n-1}$ describes the symmetries that map vertices of $\Delta_{n-1} \times \Delta_{1}$ back to, $\Delta_{n-1} \times \Delta_{1}$ while preserving adjacency and keeping $\left\{p_{1}, \ldots, p_{d-1}\right\}$ and $\left\{p_{1}^{\prime}, \ldots, p_{d-1}^{\prime}\right\}$ on the same sides of $\Delta_{n-1} \times \Delta_{1}$.
This group has order $(n-1)$ !, therefore all triangulations of $\Delta_{n-1} \times \Delta_{1}$ are symmetric, therefore they are all regular.

