

(a) A simplex is 2 neighborly so any 2 vertices form an edge, therefore the diameter is 1. $n - d > 1$ for all $d > 0$ and so the Hirsch conjecture holds.

(b) C_d is formed by creating two copies of C_{d-1} and connecting the corresponding vertices. I will show by induction that the diameter of the d -cube is d . $\Delta C_1 = 1$ and $\Delta C_2 = 2$. Assume that $C_d = d$ consider C_{d+1} . Pick any two vertices on C_{d+1} . Each one lies on a copy of C_d . If they are on the same copy then the distance between them on the graph will be at most d . If they lie on different copies of C_d then it will take one move to get from one copy of C_d to the other and then at most d moves to get from there to the selected vertex. Therefore the distance between them is at most $d + 1$. The number of facets of a d -dimensional cube is $2d$ and so $n - d = 2d - d = d < d + 1$ and so the Hirsch conjecture holds for the cube.

(c) The crosspolytope is formed by $\text{conv}\{\pm e_1, \dots, \pm e_d\}$. Between any two non-opposite vertices (e_i and $-e_i$ are opposites) there is an edge. Take $(e_j + e_i) \cdot v = 2$ where v is a vertex of the polytope. Because each vertex is some e_k and because

$e_k \cdot e_n = 0$ for $n \neq k$, and $e_k \cdot -e_k = -1$, every non opposite pair of $\pm e_i, \pm e_j$ will maximize $(e_i + e_j) \cdot v = 2$ and so every non opposite pair forms an edge. To get from one vertex to another it will take at most two moves, one to change the position of the 1, and one to change signs. Therefore the diameter of the crosspolytope is 2. In the case of \diamond_1 however there is only one possible move, a sign change and so the diameter of it is 1. The number of facets of the d dimensional crosspolytope is 2^d . $2^d - d \geq 2$ for all d greater than 1 and in that case, $2 - 1 = 1$ and so the Hirsch conjecture holds for the crosspolytope.

(d) Every two vertices of the dodecahedron are either on adjacent pentagons (facets) or they each on an edge each of whose other vertex lie on the same "central" pentagon. Because the diameter of the pentagon is 2, it will take at most two moves to get to the adjacent pentagon and then 2 more to get to the chosen vertex (in the first case) or it will take 1 move to get to the "central" pentagon, 2 moves to get to the vertex that shares an edge with the chosen vertex and then one move to get to the chosen vertex for a total of 5 moves. Therefore the diameter is 5. The dodecahedron has 12 facets, $12 - 3 = 9 > 5$ and so the Hirsch conjecture holds.

(e) Every facet of the icosahedron is a triangle and every vertex has degree 5 so by making one move along the graph you arrive at a vertex of degree 5 that only shares 2 facets with the starting vertex. By making one other move you can arrive at a vertex that shares none of the facets with the starting vertex. Because there are five choices for the first move and four non-backtracking choices for the second move, after the second move you can arrive at any desired facet. Because the facet is a triangle it will take at most one more move to reach the desired vertex. Therefore the diameter is 3. The icosahedron has 20 facets, $20 - 3 = 17 > 3$ and so the Hirsch conjecture holds.