Problem 5 (I worked in this problem with Federico Castillo and Diego Cifuentes):

Let $m_1 = (a_{11}, a_{12}, ..., a_{1n}), m_2 = (a_{21}, a_{22}, ..., a_{2n}), ..., m_n = (a_{n1}, a_{n2}, ..., a_{nn})$ be the vectors of preferences of the males,i.e, a_{ij} represents the amount of money that the male i is willing to pay (or must we pay) to date with the female j. Similarly, define $f_1 = (b_{11}, b_{12}, ..., b_{1n}), f_2 = (b_{21}, b_{22}, ..., b_{2n}), ..., f_n = (b_{n1}, b_{n2}, ..., b_{nn})$ the vectors of preferences of the females, i.e, b_{ij} represents the amount of money that the female i is willing to pay (or must we pay) to date with the male j. Using the previos notation, and knowing that we must arrange n disjoint dates with the objetive of maximize profit, what we are looking for is: $max_{\sigma\in S_n} \sum_{i=1}^{n} a_{i\sigma(i)} + b_{\sigma(i)i}$. Let $c \in \mathbb{R}^{n^2}$,

$c = (a_{11}+b_{11}, a_{12}+b_{21}, ..., a_{1n}+b_{n1}, a_{21}+b_{12}, a_{22}+b_{22}, ..., a_{2n}+b_{n2}, ..., a_{n1}+b_{1n}, a_{n2}+b_{2n}, ..., a_{nn}+b_{nn})$, (i.e., $(c)_{n(i-1)+j} = a_{ij} + b_{ji}$ for all $i, j \in \{1, 2, ..., n\}$). Now for each $\sigma \in S_n$ let $v_\sigma \in \mathbb{R}^{n^2}$ be defined as follows: $(v_\sigma)_{n(i-1)+j} = 1$ if $\sigma(i) = j$ and $(v_\sigma)_{n(i-1)+j} = 0$ otherwise, that is, in the first n entries I set 1 in the $\sigma(1)$ position and 0 in the rest, in the second n entries I set 1 in the $\sigma(2)$ position and 0 in the rest,..., in the last n entries I set 1 in the $\sigma(n)$ position and 0 in the rest. Using the previous notation the problem can be written as $max_{\{v_\sigma:\sigma\in S_n\}} c^t v_\sigma$. Defining the polytope $P = convex\{v_\sigma : \sigma \in S_n\}$($V$-description) and knowing that the max of a linear functional in a polytope always is attained at least in a vertex, the original problem is equivalent to $max_{\{x\in P\}} c^t x$.

Now lets find the $H - description$ of P. To simplify the situation, lets consider the vectors $x \in \mathbb{R}^{n^2}$ as $n \times n$ real matrices, where the first n coordinates of x are represented at the first row of the matrix,the second n coordinates are in the second row of the matrix, and so on. Using this represesentation we can observe that the vertices of P are the permutation matrices. Now I am going to prove that P (i.e the convex hull of the permutation matrices) is the set of doubly stochastic matrices. The set of doubly stochastic matrices can be viewed as the polytope in the space of $n \times n$ matrices given by the restrictions:

i) The entries of M must be nonnegative.

ii) The sum of the eantries at each row is 1.

iii) The sum of the eantries at each colum is 1.

The permutation matrices are vertices of this polytope, since if we regard them as vectors again, each of them is the only maximun of the linear functional associated to a vector that is equal to themselves (for instance, (010100001) is a vertex since it is the only maximun asociated with the functional given by $c = (010100001)$).

Now lets check that this polytope can not have more vertices:

Let M be a doubly stochastic matrix different to any permutation. Then we can find an entry m_1 of M such that $m_1 \in (0,1)$. Let m_2 be an entry in the same row of m_1 such that $m_2 \in (0,1)$. Let m_3 be an entry in the same column of m_2 such that $m_3 \in (0,1)$. We can continue in this way, going through rows and columns in consecutive steps, and taking entries with values in $(0,1)$. Since there are finitely many entries we must repeat entries in this proccess, asume WLOG that m_1 is the first entry we repeat in the procces, and we return to it at step n, i.e, $m_n = m_1$. Define $\epsilon = min\{m_i, 1 - m_i : i = 1, ..., n\}$, so $0 \leq m_i - \epsilon < m_i < m_i + \epsilon \leq 1$ for $i = 1, 2, ..., n$. We must consider two cases according to the value of n:

$*$ If n is odd ,define M^+ from M by changing the entries $m_1, m_2, ..., m_{n-1}$ to $m_1 = m_1 + \epsilon$, $m_2 = m_2 - \epsilon$,$m_3 = m_3 + \epsilon$,...,$m_{n-1} = m_{n-1} - \epsilon$, and the other entries of M remain equal. Observe that M^+ is doubly stochastic since all the entries are nonnegative (by the election of ϵ) and the sum in rows and columns is not changed. Now define M^- from M by changing the entries $m_1, m_2, ..., m_{n-1}$ to $m_1 = m_1 - \epsilon$,$m_2 = m_2 + \epsilon$,$m_3 = m_3 - \epsilon$,...,$m_{n-1} = m_{n-1} - \epsilon$ and the other entries remain equal, so M^- is doubly stochastic. Since $M = \frac{1}{2}M^+ + \frac{1}{2}M^-$ we conclude that M is not a vertex of the polytope of doubly stochastic matrices.

$*$ If n is even we get an analogous result to the previos one: setting M^+ from M by changing the entries $m_2, m_3, ..., m_{n-1}$ to $m_2 = m_2 + \epsilon$,$m_3 = m_3 - \epsilon$,...,$m_{n-1} = m_{n-1} - \epsilon$ and M^- from M by changing the entries $m_2, ..., m_{n-1}$ to $m_2 = m_2 - \epsilon$,$m_3 = m_3 + \epsilon$,...,$m_{n-1} = m_{n-1} + \epsilon$, we get two DS, such that $M = \frac{1}{2}M^+ + \frac{1}{2}M^-$ so we conclude that M is not a vertex of the polytope of doubly stochastic matrices.

Therefore if M is any dobly stochastic matrix different to a permutation matrix,it can not be a vertiex of the polytope of doubly stochastic matrices . So we conclude that $conv\{M_\sigma : M_\sigma \text{is a permutation matrix }\} =$ doubly stochastic matrices.