5. Enumerate the men and the women from 1 to n. We have $n!$ possible pairing and each pair induces a distinct permutation $\pi \in S_n$ where $\pi(i) = j$ if m_i and w_j are paired. Now S_n has a canonical representation as a group of matrices in $\mathcal{M}_n(\mathbb{R})$ given by $a_{ij} = \delta_{\pi(i)j}$, where δ is the kronecker function. Denote this matrix by A_π and let $\mathcal{A} := \{A_\pi \mid \pi \in S_n\}$. Let $P = conv\mathcal{A}$. This polytope will be the one we use in the linear program. Now let $C\ in \mathcal{M}_{n\times n}$, so that C_{ij} is the profit of arranging m_i and w_j. If we view this matrix as a vector in $R^{n\times n}$, then $C \cdot A_\pi$ is the profit made by the arrangement given by the permutation π (here the product is the usual coordinate to coordinate dot product, not the matrix product). Thus what we want is to maximize C in P. The set of vertices of P is $\mathcal{A}$, because A_π is maximized by itself in P. Thus we have the V representation of P.

It remains to find the $\mathcal{H}$ representation of P. We claim that $X = (x_{ij}) \in P$ if and only if the following three conditions hold:

i. $x_{ij} \geq 0$ for $i, j \in [n]$

ii. $\displaystyle\sum_{j=1}^{n} x_{ij} = 1$ for every $i \in [n]$

iii. $\displaystyle\sum_{i=1}^{n} x_{ij} = 1$ for every $j \in [n]$

The last two conditions say that the rows and columns add up to 1. That any point in P satisfies the three conditions is obvious, because the vertices A_π satisfy them and all of them are preserved by convex combinations. To prove the other direction we first prove another result.

Proposition 1: *Let $X = (x_{ij}) \in \mathcal{M}_n(\mathbb{R})$ be a matrix that satisfies* **i.**, **ii.** *and* **iii.**. *Then it is possible to place n chess rooks on the board in such a way every rook lies in a coordinate with a positive number and no pair of rooks attack each other.*

Proof: Consider a bipartite graph G with left vertices equal to $m_1, m_2, \ldots, m_n$ and right vertices equal to $k_1, k_2, \ldots, k_n$ and where an edge is drawn from m_i to k_j if and only $x_{ij} > 0$. The condition of the roots is equivalent to say that G has a matching, thus we have to show that Hall's condition is satisfied. Let $\mathcal{B}$ be set of left vertices of size k and let $\mathcal{C}$ be the set of right vertices that are connected to some element of $\mathcal{B}$. We have to show that $|\mathcal{C}| \geq k$. Note that for a vertex m_i the sum of the numbers of numbers of the coordinates of the implied edges is 1 (it is a complete row). It follows that the sum of the number of coordinates of the edges in $\mathcal{B}$ is k. If $\mathcal{C} \leq k-1$ the the sum of the numbers of the edges entering in C is at most $k-14$ (ever column adds up to 1) and this is impossible since the implied edges are the same as for B and by condition **i**. Thus $|\mathcal{C}| \geq k$ as desired. $\square$

We proceed to show the the assertion by strong induction on the number of positive entries of X. For the base case note that there are at least n non-zero entries in the matrix, and if there are n then the matrix is A_π for some π. Assume the result is true for all matrices with less that k positive entries and let X be a matrix that satisfies the three given conditions. By the proposition we can place n non-attacking rooks on positive entries. Assume that r is the least entry under a rook. Thus we have that $X \geq rA_\pi$ where π is the permutation associated to the position of the rooks. Also $(1-r)^{-1}(X - rA_\pi)$ satisfies the three conditions and has less positive entries than X (because the entry with an r goes to zero, and no position with a zero is changed), thus $(1-r)^{-1}(X-rA_\pi)$ is in $conv\mathcal{A}$. Also $X = (1-r)((1-r)^{-1}(X-rA_\pi))+rA_\pi$ is a convex combination of elements in $\mathcal{A}$ and is therefore a member of $\mathcal{A}$, because $\mathcal{A}$ is convex.