2 Reducing the Hirsch's conjecture to the d-step conjecture, part 1

Let P be a d-polytope with n facets and assume $n < 2d$

2.1 Any two vertices lie in a common facet

Remark 2.1. *The d-simplex has $d+1$ facets. Recall that the dual of a d-simplex is a d-simplex and a d-simplex has $d+1$ vertices.*

Lemma 2.1. *Any vertex of a d-polytope lies in at least d facets*

Proof. Let f be the number of facets that contain $v \in \mathbb{R}^d$ a vertex of P. v is the intersection of the supporting hyperplanes of the facets of P that contain v. In other words v is the solution of a system of f linear equations in d unknowns. As v is zero dimensional $f \geq d$. □

Now take v_1, v_2 two distinct vertices of P and suppose they don't lie on a common facet. Then if F is a facet of P either $v_1 \in F$ or $v_2 \in F$ so we need at least $2d$ facets which contradicts the hypothesis.

2.2 $\Delta(d,n) \leq \Delta(d-1, n-1)$

Lemma 2.2. *let F be a facet of P. F contains at most $n-1$ ridges of P.*

Proof. fix F_1 a facet of P. Because of the diamond shape of the poset of P and the uniqueness of the meet, any ridge contained in F_1 is determined by a unique diamond from the $d-2$ level to the d level. This means that the well-defined map from these type of diamonds that contain F_1 to the ridges contained in F_1 is surjective. The number of such diamonds that contain F_1 are less or equal than $n-1$ as the diamonds contain 2 facets and there are $n-1$ posible pairs of facets that contain F_1.

□

Lemma 2.1 implies that for any two vertices there is a path connecting both that lies in a facet of P. This together with lemma 2.2 shows that

$$\operatorname{diam}(P) \leq \max \operatorname{diam}(\text{facet}) \leq \max_{k \leq n-1} \Delta(d-1, k)$$

Taking the maximum on the left hand side yields

$$\Delta(d,n) \leq \max_{k \leq n-1} \Delta(d-1, k)$$

Now we claim[1] $\Delta(d-1, r-1) \leq \Delta(d-1, r)$ for $r \leq n-1$. Suppose $\Delta(d-1, r-1) > \Delta(d-1, r)$ and take a d-1 polytope Q with $r-1$ facets of largest diameter. Take vertices v_1, v_2 of largest distance. If we intersect Q with a halfspace $H := \{x | c \cdot x \leq m\}$ such that it contains all vertices of

Q except v_2 we get a new d-1 polytope with one more facet than Q (it adds more vertices too) and this new polytope has the property that there exists some vertex ω in this new facet such that the distance between v_1 and ω is the same as the distance between v_1 and v_2. This happens because any edge that joins some vertex with v_2 intersects the plane $S := \{x | c \cdot x = m\}$ in only one point. Then vertices of Q that were at distance 1 from v_2 are now at distance 1, not less, from some vertex on this facet so distances between a vertex on this facet and any other vertex not on this facet are the same as the corresponding distances to v_2. So the new polytope has diameter larger or equal than Q which contradicts our hypothesis. This proves our claim and we get together with the previous result that

$$\Delta(d,n) \leq \max_{k \leq n-1} \Delta(d-1,k) = \Delta(d-1,n-1)$$

2.3 $\Delta(d,n) \leq \Delta(n-d, 2(n-d))$

Proof. We claim that for any $0 < r \leq 2d-n$ the inequality $\Delta(d,n) \leq \Delta(d-r,n-r)$ holds. $0 < 1 \leq 2d-n$ and by 2.2 $\Delta(d,n) \leq \Delta(d-1,n-1)$. Suppose that for some $0 < r < 2d-n$, $\Delta(d,n) \leq \Delta(d-r,n-r)$. Note

$$n - r < 2(d-r) = 2d - 2r \iff 2r - r < 2d - n \iff r < 2d - n$$

By hypothesis this is true and we use 2.2 to conclude that $\Delta(d,n) \leq \Delta(d-r,n-r) \leq \Delta(d-r-1,n-r-1)$ and our claim is proved.

Now take $r = 2d-n$. By our previous claim

$$\Delta(d,n) \leq \Delta(d-(2d-n), n-(2d-n)) = \Delta(n-d, 2(n-d))$$

$\square$