Proof. Let P be a d-dimensional polytope that is $\left(\lfloor \frac{d}{2} \rfloor + 1\right)$-neighborly. Suppose for a contradiction that P has a subset $V = \{v_1, \ldots, v_{d+2}\}$ consisting of $d+2$ vertices. Then, V is affinely dependent, so there is I such that $v_I \in V$ and $v_I = \sum_{i=1}^{d+2} \lambda_i v_i$ with $\sum_{i=1}^{d+2} \lambda_i = 1$. Hence, $0 = v_I - \sum_{i=1}^{d+2} \lambda_i v_i = v_I - \lambda_1 v_1 - \ldots \lambda_{d+2} v_{d+2}$ and $1 - \lambda_1 - \ldots \lambda_{d+2} = 0$. Since $\sum_{i=1}^{d+2} \lambda_i = 1$, not all λ_i are zero. Let $\lambda'_I = 1 - \lambda_I$. Then, $\lambda_1 v_1 + \cdots + \lambda'_I v_I + \ldots \lambda_{d+2} v_{d+2} = 0$ with $\lambda_1 + \cdots + \lambda'_I + \cdots + \lambda_{d+2} = 0$ and not all λ_i are zero. Hence, there are nonzero λ_j and λ_k with opposite signs. Relabeling the v_i and λ_i if necessary, let $\lambda_1, \ldots, \lambda_n$ be nonnegative (i.e ≥ 0) and $\lambda_{n+1} \ldots \lambda_{d+2}$ be nonpositive (i.e. ≤ 0). Since $\lambda_1 + \cdots + \lambda_n + \lambda_{n+1} + \ldots \lambda_{d+2} = 0$, $\lambda_1 + \cdots + \lambda_n = -\lambda_{n+1} + \cdots + \lambda_{d+2}$. Let $\Lambda = \lambda_1 + \cdots + \lambda_n$. Then, $\Lambda > 0$ because at least one of $\lambda_1, \ldots, \lambda_n$ is nonzero. Hence, $1 = \frac{1}{\Lambda} \sum_{i=1}^{n} \lambda_i = -\frac{1}{\Lambda} \sum_{i=n+1}^{d+2} \lambda_i$, so $v = \sum_{i=1}^{n} \frac{\lambda_i}{\Lambda} v_i = \sum_{i=1}^{d+2} (-\frac{\lambda_i}{\Lambda} v_i)$ is a convex combination of $v_1, \ldots, v_n$ and a convex combination of $v_{n+1}, \ldots, v_{d+2}$, so $v \in \operatorname{conv}(v_1, \ldots, v_n)$ and $v \in \operatorname{conv}(v_{n+1}, \ldots, v_{d+2})$, implying $v \in \operatorname{conv}(v_1, \ldots, v_n) \cap \operatorname{conv}(v_{n+1}, \ldots, v_{d+2})$. Let $V_1 = \{v_1, \ldots, v_n\}$ and $V_2 = \{v_{n+1}, \ldots, v_{d+2}\}$. Then, $V_1 \cup V_2 = V$ and $V_1 \cap V_2 \neq \emptyset$, and $\operatorname{conv}(V_1) \cap \operatorname{conv}(V_2) \neq \emptyset$. Hence, $V_1 \cap V_2 = \emptyset$, $|V| = |V_1| + |V_2|$, so renaming the sets if necessary, let $|V_1| \leq \lfloor \frac{d}{2} \rfloor + 1$. Since $\operatorname{conv}(V_1) \cap \operatorname{conv}(V_2) \neq \emptyset$, every hyperplane H that contains V_1 and does not contain points in the interior of P, contains points from $\operatorname{conv}(V_2)$, so H also contains at least one vertex from V_2. Then, $\operatorname{conv}(V_1)$ has vertices that are not in V_1, so $\operatorname{conv}(V_1)$ does not define a face consisting of $|V_1|$ vertices, implying that P is not $|V_1|$-neighborly, which is a contradiction to P being $\left(\lfloor \frac{d}{2} \rfloor + 1\right)$ -neighborly. Hence, P has less than $d+2$ vertices. Since P is d-dimensional, P has $d+1$ vertices, so P is a simplex. Therefore, if a d-dimensional polytope is $\left(\lfloor \frac{d}{2} \rfloor + 1\right)$ - neighborly, then it is a simplex.

$\square$