Problem 4

Note that we are excluding the empty set from this bijection. Suppose $a = (a_1, ..., a_n) \in \mathbb{R}^n$. Let $b_1 = min\{a_i : 1 \leq i \leq n\}$ and for $k > 1$, suppose that $\{a_i : a_i > b_{k-1}, 1 \leq i \leq n\} \neq \emptyset$, then let $b_k = min\{a_i : a_i > b_{k-1}, 1 \leq i \leq n\}$. Then $\exists s \leq n$, such that s is the maximal number such that $\{a_i : a_i > b_{s-1}, 1 \leq i \leq n\} \neq \emptyset$. So for $1 \leq k \leq s$, let $C_k = \{i : a_i = b_k\}$. Then the face in Π_{n-1} that maximizes a is completely dependent on the sets $C_1, ..., C_s$.

Proof: Let F be the face of Π_{n-1} that maximizes a, and let $v = (v_1, ..., v_n)$ be a vertex of F. Let $|C_i| = c_i$ for all i. Then we will show that $\forall j \in C_1$, $v_j \in \{1, 2, ..., c_1\}$. Suppose $v_j \notin \{1, 2, ..., c_1\}$, then $v_j > c_1$ and $\exists k \notin C_1$ with $v_k \in \{1, 2, ..., c_1\}$. Let $w = (w_1, ..., w_n)$ be the vertex of P that is identical to v except the j^{th} and k^{th} coordinate is switched. In other words, $w_j = v_k$, $w_k = v_j$ and $w_i = v_i$ for $i \neq j, k$. Then

$$a \cdot w - a \cdot v = a_k(w_k - v_k) + a_j(w_j - v_j) = a_k(v_j - v_k) + a_j(v_k - v_j) = (v_j - v_k)(a_k - a_j) > 0,$$

since $v_j > v_k$ and $a_k > a_j$. Thus, $a \cdot w > a \cdot v$ so v does not maximize a. A contradiction.

Similarly, now that we have shown $j \in C_1$, implies $v_j \in \{1, 2, ..., c_1\}$, the same proof will give us $j \in C_2$ implies $v_j \in \{c_1 + 1, c_1 + 2, ..., c_1 + c_2\}$, and more generally $j \in C_i$ implies $v_j \in \{c_1 + c_2 + ... + c_{i-1} + 1, c_1 + c_2 + ... + c_{i-1} + 2, ..., c_1 + c_2 + ... + c_{i-1} + c_i\}$. It is also clear that if u is another vertex that satisfies $j \in C_i$ implies $u_j \in \{c_1 + c_2 + ... + c_{i-1} + 1, c_1 + c_2 + ... + c_{i-1} + 2, ..., c_1 + c_2 + ... + c_{i-1} + c_i\}$, then $a \cdot u = a \cdot v$. So u maximizes a. Thus, u is a vertex in F if and only if $j \in C_i$ implies $u_j \in \{c_1 + c_2 + ... + c_{i-1} + 1, c_1 + c_2 + ... + c_{i-1} + 2, ..., c_1 + c_2 + ... + c_{i-1} + c_i\}$.

So we have a map $F \mapsto [C_1][C_2]...[C_s]$ where $[C_1][C_2]...[C_s]$ is an ordered partition of $[n]$. Call this map ϕ, we want to show that ϕ is bijective. Suppose $\phi(F) = \phi(G) = [C_1][C_2]...[C_s]$, then: v is a vertex of $F \Leftrightarrow j \in C_i$ implies $v_j \in \{c_1 + c_2 + ... + c_{i-1} + 1, c_1 + c_2 + ... + c_{i-1} + 2, ..., c_1 + c_2 + ... + c_{i-1} + c_i\} \Leftrightarrow v$ is a vertex of G. So ϕ is injective.

Now let $[D_1][D_2]...[D_r]$ be an ordered partition of $[n]$. Let $x = (x_1, ..., x_n) \in \mathbb{R}^n$, where if $j \in D_i$ then $x_j = i$. Now there exist a face, F, that maximizes x. Constructing $C_1, ..., C_s$ for F (as we did in the first paragraph of this problem), we see trivially that $r = s$, and $D_i = C_i$ for $1 \leq i \leq s$. Thus, $\phi(F) = [D_1][D_2]...[D_r]$. We still need to show that F is not the empty set. Let $|D_i| = d_i$. Let φ_i be any bijection between D_i and $\{d_1 + d_2 + ... + d_{i-1} + 1, d_1 + d_2 + ... + d_{i-1} + 2, ..., d_1 + d_2 + ... + d_{i-1} + d_i\}$. Let $y = (y_1, ..., y_n) \in \mathbb{R}^n$ such that if $j \in D_i$ then $y_j = \varphi_i(j)$. Then y permutes the elements of $[n]$, so y is a vertex of Π_{n-1}. Also, $j \in D_i$ implies $y_j \in \{d_1 + d_2 + ... + d_{i-1} + 1, d_1 + d_2 + ... + d_{i-1} + 2, ..., d_1 + d_2 + ... + d_{i-1} + d_i\}$. So y maximizes x. Hence F is not empty, and so ϕ is surjective.

Finally, we want to show that is $F \mapsto [C_1][C_2]...[C_s]$ then $s = n - dim(F)$. We will call $[D_1][D_2]...[D_r]$ a refinement of $[C_1][C_2]...[C_s]$, if $\forall i$, $\exists j$ such that $D_i \subseteq C_j$ and if $D_i \subseteq C_j$ and $D_{i'} \subseteq C_{j'}$ then $i < i'$ implies that $j \leq j'$. Now suppose F and G are faces with $\phi(F) = [C_1][C_2]...[C_s]$ and $\phi(G) = [D_1][D_2]...[D_r]$ and $[D_1][D_2]...[D_r]$ is a refinement of $[C_1][C_2]...[C_s]$. Then, there exists $0 = n_0 < n_1 < ... < n_s = n$ such that

$$C_i = \bigcup_{j=n_{i-1}+1}^{n_i} D_j.$$

Let $|D_i| = d_i$ and $|C_i| = c_i$ and let v be a vertex of G. Then $j \in D_i$ implies $v_j \in \{d_1 + d_2 + ... + d_{i-1} + 1, d_1 + d_2 + ... + d_{i-1} + 2, ..., d_1 + d_2 + ... + d_{i-1} + d_i\}$, and $n_{k-1} + 1 \leq i \leq n_k$ for some

$1 \leq k \leq s$. This implies that $v_j \in \{n_{k-1}+1, n_{k-1}+2, ..., n_k\}$. So v is a vertex of F. Thus $G \subset F$. Since ϕ is injective, we know that $F \neq G$ (assuming $[C_1][C_2]...[C_s] \neq [D_1][D_2]...[D_r]$) so G is a subface of F. However, if $\mathfrak{C}_k$ is a ordered partition of $[n]$ into $n-k$ parts, then there exists ordered partitions of $[n]$, $\mathfrak{C}_0, \mathfrak{C}_1, ..., \mathfrak{C}_{k-1}, \mathfrak{C}_{k+1}, ..., \mathfrak{C}_{n-1}$, such that $\mathfrak{C}_i$ partitions $[n]$ into $n-i$ parts and $\mathfrak{C}_i$ is a refinement of $\mathfrak{C}_{i-1}$. Suppose $\forall i$, F_i is a face with $\phi(F_i) = \mathfrak{C}_{n-i}$. Then we have $F_0 \subset F_1 \subset ... \subset F_n$. However, since this is a chain of n pairwise disjoint faces, none of which are the empty set, then this implies that $dim(F_i) = i$. So our proof is complete.