3. (a) $f_{\mathrm{pyr}(P)} = (1+x)f_P(x)+1$. We assume that $P \subset \mathbb{R}^e$ and for the construction of P we embed it into $x_{e+1} = 0$. Recall that a face of a polytope is the convex hull of the vertices of P it contains. Let $\mathbf{v}$ be the vertex added to obtain $\mathrm{pyr}(P)$ and let F be a face of $\mathrm{pyr}(P)$. We study two cases:

- If $\mathbf{v} \notin F$ we claim that F is a face of P. All the vertices of F lie in P. Let c be a functional that is maximized by F. Every point on P has it's last coordinate equal to zero, so we can omit the last coordinate of c and view it as a functional in $\mathbb{R}^e$ acting on P. This shows that F must be a face of P since the values of c and the new functional coincide in all of P.
- If $\mathbf{v} \in F$ we claim that the convex hull of the vertices of F different from $\mathbf{v}$ are a face of P. We can assume that $\mathbf{v} \in \mathrm{span}\mathbf{e}_{e+1}$ by translating the whole $\mathbb{R}^{e-1}$ (this gives an isometry, thus the combinatorial structure of P does not change, because the geometry of the figure does not change at all). Now let $\mathbf{c}$ be vector such that $c \cdot \_$ is maximized by F. Write $\mathbf{c} = \mathbf{c}' + \mathbf{c}^{\perp}$, where $\mathbf{c}'$ is in $x_{e+1} = 0$ and $\mathbf{c}^{\perp} \in \mathrm{span}(\mathbf{e}_{e+1})$. Then for $\mathbf{u}$ a vertex in P we have that $\mathbf{c} \cdot \mathbf{u} = \mathbf{c}' \cdot \mathbf{u}$ and $\mathbf{c} \cdot \mathbf{v} = \mathbf{c}^{\perp} \cdot \mathbf{v}$. Thus $\mathbf{c}'$ maximizes exactly the vertices of F in P and we can view it as a vector in $\mathbb{R}^e$ thus if we take the vertices of F that are different from $\mathbf{v}$, because $\mathbf{c}'$ maximizes them.

We showed that a face of $\mathrm{pyr}(P)$ is either a face of P or a pyramid of a face of P. The dimension of a face increases by one if we take the pyramid. Thus - if $d \geq 1$ - the d-dimensional faces of $\mathrm{pyr}(P)$ are given by $f_d + f_{d-1}$ where f_k denotes the k dimensional faces of P. Since we added exactly one vertex we get that $f_{\mathrm{pyr}(P)}(x) = 1 + f_P(x) + x f_P(x) = f_P(x)(x+1) + 1$.

(b) $f_{P\times Q}(x) = f_P(x) f_Q(x)$. We claim that F is a face of $P \times Q$ if and only if F is of the form $G \times H$ where G is a face of P and H is a face of Q. If this is true then we are done, because the dimension of a product is the sum of the dimensions. Thus the answer follows immediately if we prove that the classification is well done.
If c is a linear functional in the space of $P \times Q$, then we can view it as (c_p, c_q) where c_p is in the space of P and c_q is in the space of Q. Thus the elements that maximize c y $P \times Q$ come from some element that maximizes c_p in the first coordinate and some element that maximizes c_q in Q, that is, the face is the product of two faces.
If G and H are faces of P and Q respectively let c_G and c_H be functionals maximized by G and H respectively. Then (c_G, c_H) is maximized by $G \times H$, since the coordinates in c_G only act on the part that comes from G and the same for H (it is a direct dot product).