6. (a) If $P, Q \subset \mathbb{R}^d$ are polytopes, prove that $P \cap Q$ is also a polytope.

Proof. Since P and Q can be satisfied by a system in inequalities, the intersection is the set of points that satisfy all the inequalities in P and Q. It is bounded because both P and Q are bounded. Therefore it is a polytope. $\square$

(b) If $P, Q \subset \mathbb{R}^d$ are polytopes, prove that $P + Q$ is also a polytope.

Proof. Let $P = \operatorname{conv}(p_1, \cdots, p_n)$ and $Q = \operatorname{conv}(q_1, \cdots, q_m)$. Define the following set: $S = \operatorname{conv}(p_i + q_j | i \in [n] \text{ and } j \in [m])$ I will show that $P + Q = S$. To show $P + Q \supseteq S$, let $x \in S$ be given. Then you can write x in the following way: There exist γ_{ij}s such that $\sum \gamma_{ij} = 1$ and each $\gamma_{ij} \geq 0$ so that:

$$x = \sum_{i=1}^{n} \sum_{j=1}^{m} \gamma_{ij}(p_i + q_j)$$

$$x = \sum_{i=1}^{n} \sum_{j=1}^{m} \gamma_{ij} p_i + \gamma_{ij} q_j$$

$$x = \sum_{i=1}^{n} \sum_{j=1}^{m} \gamma_{ij} p_i + \sum_{i=1}^{n} \sum_{j=1}^{m} \gamma_{ij} q_j$$

$$x = \sum_{i=1}^{n} p_i \sum_{j=1}^{m} \gamma_{ij} + \sum_{j=1}^{m} q_j \sum_{i=1}^{n} \gamma_{ij}$$

Notice that if we define $\sum_{i=1}^{n} \gamma_{ij} = \mu_j$ and $\sum_{j=1}^{m} \gamma_{ij} = \lambda_i$, then we have (by construction) that $\sum \lambda_i = \sum \mu_j = 1$ and $\lambda_i, \mu_j \geq 0$. We are left with:

$$x = \sum_{i=1}^{n} \lambda_i p_i + \sum_{j=1}^{m} \mu_j q_j$$

Thus, x is a convex combination of vertices of P plus a convex combination of vertices in Q. In other words, we have $x = p + q$ for some $p \in P$ and $q \in Q$; therefore $S \subseteq P + Q$.
Now to show $P + Q \subseteq S$, let $x \in P + Q$ be given. Then there exist $p \in P$ and $q \in Q$ such that $x = p + q$. We can also write p and q as convex combinations of vertices from their respective sets. so $x = \sum p_i \lambda_i + \sum q_j \mu_j$ we can break up λ_i into the sum of m terms such,

i.e. $\lambda_i = \sum_{j=1}^{m} \gamma_{ij}$. Similarly, you can break up μ_j into the sum of n terms so that $\sum_{i=1}^{n} \gamma_{ij}$. You are then left with:

$$x = \sum_{i=1}^{n} p_i \sum_{j=1}^{m} \gamma_{ij} + \sum_{j=1}^{m} q_j \sum_{i=1}^{n} \gamma_{ij}$$

Which can lead back to:

$$x = \sum_{i=1}^{n} \sum_{j=1}^{m} \gamma_{ij}(p_i + q_j)$$

Again, by construction, $\sum_i \sum_j \gamma_{ij} = 1$ and $\gamma_{ij} \geq 0$, so it is a convex combination of vertices of the form $p_i + q_j$. So, $x \in S$; therefore, $P + Q \subseteq S$. $\square$

(c) If $P \subset \mathbb{R}^d$ and $Q \in \mathbb{R}^e$ are polytopes, prove that $P \times Q$ is also a polytope.

Proof. Consider the H-description of polytopes. Let $P = \left\{x \in \mathbb{R}^d : Ax \leq \alpha\right\}$ and $Q = \left\{y \in \mathbb{R}^d : By \leq \beta\right\}$. Let $S = \left\{\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^{d+e} : \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \leq \begin{bmatrix} \alpha \\ \beta \end{bmatrix}\right\}$ Notice, you can $S = \left\{\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^{d+e} : \begin{bmatrix} Ax \\ By \end{bmatrix} \leq \begin{bmatrix} \alpha \\ \beta \end{bmatrix}\right\} = \left\{\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^{d+e} : x \in P \text{ and } y \in Q\right\} = P \times Q$ $\square$