5. (a) ***Lemma 1:*** *Let $P \subset \mathbb{R}^d$ be a d dimensional polytope with vertices $v_1, , v_2, \dots, v_t$. A point $x \in P$ is interior if and only if cone$\{v_i - x\} = \mathbb{R}^d$.*

Proof: We may assume that $x = 0$ adding $-x$ to P.

$\Rightarrow$) Assume first that 0 is interior. Then there is an open ball $B_r(0)$ that is completely contained in P. Now let $v \in \mathbb{R}^d$ and let k be a positive real number such that $\|kv\| < r$. Then $kv \in B_r(0) \subset P$, so $kv = \sum \lambda_i v_i$ for some non negative lambda such that $\sum \lambda_i = 1$. It follows that $v = \sum k^{-1} \lambda_i v_i$ and we are done since $k^{-1}\lambda_i \geq 0$ for all i.

$\Leftarrow$) Assume that x is not an interior point. Then x is satisfies an equality of the $\mathcal{H}$-Polytope equations (otherwise it is in the intersection of the open half planes). Take a normal vector v to hyperplane at 0 is in, that satisfies the opposite inequality. All the positive multiples of v do not belong P, because the intersection of span$\{v\}$ and the hyperplane is unique and 0 is in the intersection. Thus it cannot happen that $v \in \text{cone}\{v_i\}$, because if $\sum \lambda_i v_i = v$, with $\lambda_i \geq 0$, then we can divide by the sum of the λ_i and get a positive multiple of v that is a convex combination of $\{v_i\}$ and is therefore in P. It follows that cone$\{v_i\} \neq \mathbb{R}^d$.

Lemma 2: *Let $B \subseteq \mathbb{R}^d$ be a finite set of vectors that contains a basis $b_1, b_2, \ldots, b_d$ of $\mathbb{R}^d$ and such that $-\sum b_i \in cone(B)$. Then $cone(B) = \mathbb{R}^d$*

Proof: Let $y \in \mathbb{R}^d$. Since $\{b_i\}$ is a basis there are real numbers r_i such that $y = \sum r_i b_i$. If r_i is non negative for all i then $y \in \text{cone}\{b_i\} \subseteq \text{cone}(B)$. If not assume without loss of generality that r_1 is the smallest coefficient (thus $r_1 < 0$). Then we can write

$$y = |r_1|(-\sum b_i) + \sum_{i=2}^{d}(r_i + |r_1|)b_i$$

This finishes the problem because $r_i + |r_1| \geq r_1 + |r_1| = 0$ and the postive combinations of elements of the cone is in the cone.

We proceed to show the interior Cartheodory theorem. We can assume again that the interior point we are taking is equal to 0, translating P. Let $v_1, v_2, \ldots, v_t$ be the vertices of P. The fact that 0 is interior yields that cone$\{v_i\} = \mathbb{R}^d$ by lemma 1. Now the set $\{v_i\}$ contains a basis of $\mathbb{R}^d$, because $\mathbb{R} = \text{cone}\{v_i\} \subseteq \text{span}v_i$. Assume without loss of generality that $\mathcal{B} = \{v_1, v_2, \ldots v_d\}$ is a basis. Now consider the vector $-\sum_{i=1}^{d} v_i$. We know that it is contained in cone$\{v_i\}_{i=1}^{t}$, so by Caratheodory for cones, there is a subset $\mathcal{A} \subseteq \{v_i\}_{i=i}^{t}$, such that $|\mathcal{A}| \leq d$ and $-\sum_{i=1}^{d} v_i \in \text{cone}(\mathcal{A})$. Take $B = \mathcal{B} \cup \mathcal{A}$ and note that B satisfies the conditions of lemma 2 taking $\mathcal{B}$ as the ground basis. Then cone$(B) = \mathbb{R}^d$. Thus by lemma 1 0 is an interior point of B. Now $|B| \leq |\mathcal{A}| + |\mathcal{B}| \leq d + d = 2d$, so we proved what we wanted.

(b) Take the cross polytope, that is $P = \text{conv}\{\pm e_i\}$, where $e_1, e_2, \ldots, e_d$ is the standard basis of $\mathbb{R}^d$. Thes polyhedron has $2d$ vertices. and contains the origin in it's interior. If we remove some vertex v thenthe intersection of the half plane $v \cdot x > 0$ and the new polyhedron P' is empty (this is obvious, because $v = \pm e_i$ for some i and it is the only vector having that sign on coordinate i). Now 0 is a limit point of $v \cdot x > 0$ (put a sequence that converges to 0 and has the same sign as $(v)_i$ in the i-th coordinate), thus 0 can't be an interior point of P'. It follows that $2d$ can't be reduced.

(c) I claim that the only polytopes P for which $2d$ is the lowest bound are the crossed polytopes with concurrent diagonals, and the only point that fails is the intersection of the diagonals. This cases can't be lowered and the proof is the same as in (b) by making a change of basis (the negative vectors of the basis may be multiplied by a positive scalar).

To prove the result note that $-\sum b_i$ of lemma 2 can be replaced by any point of the form $u = -\sum \lambda_i b_i$ where $\lambda_i > 0$ for all i (the proof is the same, take a large enough positive multiple of u and fix the coefficients with the basis). Assume that P is a polytope for which the $2d$ can't be reduced for the point x. We may translate P and assume that $x = 0$. Take a basis of vectors with vertices in P, and pick a set $\mathcal{A}$ as in (a). Then it must be that $|\mathcal{A}| = d$ also $\mathcal{A}$ is a basis of $\mathbb{R}$ because otherwise it lies in a $d-1$ dimensional subspace and by caratheodory we can choose $d-1$ of those vectors that contain $u_B = -\sum b_i$. Note that the coefficients of U_B in the cone are all positive or we can remove the vector with zero coefficient. The same happens with any point in the interior of cone$(-\mathcal{B})$ by the extension of the lemma 2. Thus cone$(-\mathcal{B}) \subseteq$ cone$(\mathcal{A})$. Now we can reverse the roles of $\mathcal{A}$ and $\mathcal{B}$, in the previous argument and we get that cone$(-\mathcal{A}) \subseteq$ cone$\mathcal{B}$. But cone$(-A) = -$cone(A) for any finite set of vectors A, thus we conclude that cone$(\mathcal{A}) =$ cone$(-\mathcal{B})$. Now we claim that $\mathcal{A}$ is the same as $-\mathcal{B}$ up to multiplication by positive scalars. If not, then some $a \in A$ is equal to $\sum \lambda_i b_i$ with $\lambda_i \leq 0$ and two non zero coefficients. Assume WLOG that they are λ_1 and λ_2. Then there are vectors $u_3, u_4, \ldots, u_d$ in $\mathcal{A}$ such that the coefficient of b_i is negative for u_i but then, $a + \sum u_i$ is a positive combination of $-\mathcal{B}$ thus we could reduce one vector.

Now it remains to show that there are no more vertices in P. If there is another vertex v it can't be in cone$(\mathcal{B})$ neither in cone$(-\mathcal{B})$ (otherwise we reduce, because a line contains at most two vertices, so it is not a multiple of an element of the basis). Thus $v = \sum a_i b_i$ where $\min a_i < 0 < \max a_i$. Assume WLOG that $a_1 < 0 < a_2$. Then cone$(((\mathcal{A} \cup \mathcal{B}) \backslash \{b_2, a_1 b_1\}) \cup \{v\}) = \mathbb{R}^d$ where $a_1 b_1$ is the multiple of b_1 in $\mathcal{A}$, because we can multiply v by a large enough constant and fix the coefficients with the rest of the basis. We are done.