Problem 1.1 (*f*-vectors of 3-polytopes) Let v, e, and f be integers. Prove that there exists a polytope with v vertices, e edges, and f faces if and only if:

$$v - e + f = 2; v \le 2f - 4, and f \le 2v - 4.$$

You may assume Euler's formula (v - e + f = 2); we will prove it later in class.

Proof of \Rightarrow :

Since every face has at least 3 edges, and every edge is the intersection of exactly 2 faces, we know that

$$e \geq \frac{3f}{2}$$

rearranging and plugging Euler's formula into this gives:

$$3f \le 2e \Rightarrow$$

 $3f \le 2v + 2f - 4 \Rightarrow$
 $f \le 2v - 4$ (1)

similarly, since every vertex has valence (or degree) at least 3, and every edge connects exactly 2 faces, we know that

$$e \geq \frac{3v}{2}$$

and just as above

$$3f \le 2e \Rightarrow$$

 $3f \le 2v + 2f - 4 \Rightarrow$

$$f \le 2v - 4 \quad (2)$$

so any polytope of dimension 3 that does exist must satisfy the formulas. \blacksquare

Proof of \Leftarrow :

we wish to construct a three dimensional polytope for every lattice point in the figure below:



Here we have faces plotted against vertices, subject to our constraints. The so-called *pyramid line* above refers to the points that are realizable as regular pyramids, where the base is a regular (v - 1)-gon.

Now consider the following two *constructions*:

1) In a polytope with v vertices, e edges, and f faces, as well as a vertex of valence 3, truncate the polytope (see image) to create a new polytope with v + 2 vertices, e + 3 edges, and f + 1 faces:



2)In a polytope with v vertices, e edges, and f faces, as well as at least one triangular face, add a vertex *outside* the triangular face (see image), effectively concatenating the polytope with a tetrahedron to create a new polytope with v + 1 vertices, e + 3 edges, and f + 2 faces:



Every construction of the first type results in a vertex of valence 3, and so can be repeated indefinitely, and every construction of the second type results in more triangular faces and can also be repeated indefinitely.

The pyramids from the center line have both triangular faces and vertices of valence 3, therefore, all cases can be exhausted by the above constructions as shown below:

