# discrete geometry

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## homework four (due wed. oct. 20 at midnight)

Instructions. As last time. Please work together and write separately. State who you worked with **in each problem**. E-mail your hw to `discretegeometry@gmail.com`. LaTeX gives you 10%.

1. (Triangulations of $\Delta_{n-1} \times \Delta_1$) Describe all the triangulations of the product of simplices $\Delta_{n-1} \times \Delta_1$. Are they all regular?

2. (Eulerian numbers, part 1.) Prove that for any positive integer $d$ one can write

$$\sum_{t \geq 0} (t+1)^d z^t = \frac{A(d,1)z^0 + \cdots + A(d,d)z^{d-1}}{(1-z)^{d+1}}$$

where $A(d,0), A(d,1), \ldots, A(d,d)$ are positive integers satisfying, for all $1 \leq k \leq d$.

$$A(d,k) = (d-k+1)A(d-1,k-1) + kA(d-1,k)$$

(We write $A(d,k) = 0$ if $k \leq 0$ or if $k \geq d+1$.)

3. (Eulerian numbers, part 2.) Let $E(d,k)$ be the number of permutations of $[d]$ having exactly $k-1$ descents. Prove that, for all $1 \leq k \leq d$,

$$E(d,k) = (d-k+1)E(d-1,k-1) + kE(d-1,k)$$

Conclude that $E(d,k) = A(d,k)$ for all integers $1 \leq k \leq d$.
Also conclude that $A(d,k) = A(d, d+1-k)$ for all integers $1 \leq k \leq d$.
(We write $E(d,k) = 0$ if $k \leq 0$ or if $k \geq d+1$.)

4. (Generating functions of polynomials.) Let $f : \mathbb{N} \to \mathbb{R}$ be a function and $d \in \mathbb{N}$. Write

$$\sum_{t \geq 0} f(t) z^t = \frac{g(z)}{(1-z)^{d+1}}$$

Prove that the following are equivalent:

- $f$ is a polynomial of degree $d$
- $g$ is a polynomial of degree at most $d$ such that $g(1) \neq 0$.

5. (Ehrhart computations) Compute the Ehrhart polynomial and the Ehrhart series of the polytope $P = \mathrm{conv}\{(0,0,0), (0,0,3), (1,0,0), (1,1,0), (2,1,0), (2,0,1)\} \subset \mathbb{R}^3$.

6. (The staircase triangulation of $\Delta_{m-1} \times \Delta_{n-1}$). Let $\mathsf{e_i}$ and $\mathsf{f_j}$ be the standard unit vectors in $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively, and let $\mathsf{v_{ij}} = \mathsf{e_i} \times \mathsf{f_j} \in \mathbb{R}^{\mathsf{m+n}}$. Let

$$\Delta_{m-1} \times \Delta_{n-1} = \mathrm{conv}\{\mathsf{v}_{ij} : 1 \leq i \leq m, 1 \leq j \leq n\}.$$

Now consider the rectangular grid with corners $(1,1)$ to $(m,n)$. A *staircase* is a path from $(1,1)$ to $(m,n)$ that only steps north or east along edges of the grid. For such a staircase $S$ define a polytope

$$P_S = \mathrm{conv}\{\mathsf{v}_{ij} \,:\, (i,j) \text{ is a vertex of } S\}.$$

Prove that, as $S$ ranges over all $\binom{m+n-2}{m-1}$ staircases, the polytopes $P_S$ form a triangulation of $\Delta_{m-1} \times \Delta_{n-1}$.

7. In at most half a page, tell me your thoughts about the project. What topics are you considering? Are you interested in trying to solve an open problem / surveying the current state of a topic / understanding an interesting result / writing some useful software? Do you already have a partner? Are you interested in a partner in Bogotá / San Francisco?
(The only wrong answer is "I don't know.")