discrete geometry

federico ardila

san francisco state university
universidad de los andes

homework three (due wed. oct. 6 at midnight)

Instructions. As last time. Please work together and write separately. State who you worked with **in each problem**. E-mail your hw to `discretegeometry@gmail.com`. LaTeX gives you 10%.

1. (How neighborly can you be?) If a d-dimensional polytope is $\left(\lfloor \frac{d}{2} \rfloor + 1\right)$ - neighborly, prove that it is a simplex.

   (Recall that a polytope is k-neighborly if any set of k vertices forms a face.)

2. (The spanning tree polytope.) Let G be a graph and $ST(G)$ its spanning tree polytope.
   (a) Prove that χ_T is a vertex of $ST(G)$ for every spanning tree T of G.
   (b) Prove that two vertices χ_T, χ'_T are adjacent in $ST(G)$ if and only if there exist edges $e \in T - T'$ and $f \in T' - T$ such that $T' = T - e \cup f$.
   (c) Use (b) to prove that the diameter of $ST(G)$ is less than the number of vertices of G.

3. (Reducing Hirsch's conjecture to the d-step conjecture, part 1.) Let P be a d-polytope with n facets, and assume $n < 2d$.
   (a) Prove that any two vertices lie in a common facet.
   (b) Use (a) to conclude that $\Delta(d, n) \leq \Delta(d-1, n-1)$.
   (c) Use (b) and induction to conclude that $\Delta(d, n) \leq \Delta(n-d, 2(n-d))$.

4. (Reducing Hirsch's conjecture to the d-step conjecture, part 2.) Let P be a d-polytope with n facets, and assume $n > 2d$.
   (a) Prove that any two vertices lie away from a common facet.
   (b) Use (a) to conclude that $\Delta(d, n) \leq \Delta(d+1, n+1)$.
   (c) Use (b) and induction to conclude that $\Delta(d, n) \leq \Delta(n-d, 2(n-d))$.

5. (A linear programming example.) A heterosexual-only (!) online dating service has n male and n female members. Each member is shown the profiles of the subscribers of the opposite sex, and answers the questionnaire: how much money are you willing to pay, or how much money must we pay you, to date each member? The goal of the agency is to arrange n (disjoint) dates and make the largest profit possible.

   Phrase this as a linear programming question. Find the V-description and the H-description of the relevant polytope.

6. (Diameters of graphs of polytopes.) Find the diameters of the graphs of:
   (a) the simplex Δ_{d-1},
   (b) the cube C_d,
   (c) the crosspolytope $\diamond_d$,
   (d) the dodecahedron, and
   (e) the icosahedron.

   Verify that the Hirsch conjecture holds for all of them.