discrete geometry

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homework two (due wed. sep. 22 at midnight)

Instructions. As last time. Please work together and write separately. State who you worked with **in each problem**. E-mail your hw to `discretegeometry@gmail.com`. LaTeX gives you 10%.

1. (Putting polytopes in a positive orthant.) Prove that for every polytope P one can find an integer m and an affine subspace $A \subset \mathbb{R}^m$ such that P is affinely isomorphic to $A \cap \mathbb{R}^m_{\geq 0}$.
2. (3-polytopes with few edges) Describe all the possible combinatorial types of 3-polytopes with at most 9 edges. Which pairs of types are polar to each other?
3. (f-polynomials of prisms and products)
   (a) Find the f-polynomial of the pyramid $\mathrm{pyr}(P)$ over a polytope P in terms of the f-polynomial of P.
   (b) Find the f-polynomial of the product $P \times Q$ of two polytopes P and Q in terms of the f-polynomials of P and Q.
4. (Faces of permutahedra) Prove that the k-dimensional faces of the permutahedron Π_{n-1} are in bijection with the ordered partitions of $[n]$ into $n-k$ parts; that is, the ways of writing the set $[n] := \{1, \ldots, n\}$ as an ordered, pairwise disjoint union $[n] = A_1 \cup \cdots \cup A_k$ of non-empty subsets of $[n]$.
5. (Shuffling flags) A *flag* of faces of a d-polytope P is a collection $\mathcal{F} = (F(-1), F(0), \ldots, F(d))$ of faces of P such that $\emptyset = F(-1) \subset F(0) \subset \cdots \subset F(d) = P$ and $\dim F(i) = i$ for $0 \leq i \leq d$. Let $\mathsf{Flags}(P)$ be the set of such flags.

   Given a flag of faces $\mathcal{F} \in \mathsf{Flags}(P)$, define $T\mathcal{F} = (TF(-1), TF(0), \ldots, TF(d))$ recursively by $TF(-1) = \emptyset$, $TF(d) = P$ and, for each $i \geq 0$, $TF(i)$ is the unique i-dimensional face other than $F(i)$ which contains $TF(i-1)$ and is contained in $F(i+1)$.
   (a) Prove that T is a bijection from $\mathsf{Flags}(P)$ to itself. Construct T^{-1} explicitly.
   (b) Prove that if $0 \leq r < s \leq d$ then $T^r\mathcal{F}(k) \neq T^s\mathcal{F}(k)$ for all k with $0 \leq k \leq d-1$.
   (c) (Open problem – extra credit.) What else can you say about T?
6. (Many different 0-1 polytopes) A 0-1 polytope is one such that every coordinate of every vertex equals 0 or 1. Let $f(d)$ be the number of combinatorial types of d-dimensional 0-1 polytopes.
   (a) Prove that $2^{2^{d-2}} < f(d) < 2^{2^d}$.
   (b) (Open problem - extra credit.) Can you improve this bound significantly?

As always, I am happy to give you hints on any of the problems, particularly on 5 and 6. In fact, here is a hint for 6(a):

(Consider the set S of 0-1 polytopes in $\mathbb{R}^d$ which contain $(0,0,\ldots,0,0), (1,1,\ldots,1,0)$ and $(a_1, \ldots, a_{d-1}, 1)$ for all $a_i \in \{0,1\}$, and which do not contain $(1,0,\ldots,0,0)$ and $(0,1,\ldots,1,0)$. How many polytopes are in the set S? If you partition S into combinatorial equivalence classes, how large can a class be?)