discrete geometry

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homework one (due mon. sep. 6 (sf) or wed. sep. 8 (bog))

Note. You are strongly encouraged to work together on the homework, but you **must** state who you worked with. You **must** also write your solutions independently and in your own words. You may turn in your homework at the end of the class, or by email **in a single .pdf file** before midnight at `discretegeometry@gmail.com`. (No exceptions.)

Grading. The homework is graded out of 50 points. Each problem is worth 10 points. (I know...)

LaTeX incentive. You will receive a 10% increase on your grade if you turn in your homework as a .pdf file produced in LaTeX. (Other software doesn't count, sorry.) I reserve the right to ask you to type up your subsequent homework in LaTeX if I cannot understand your handwriting.

1. (f-vectors of 3-polytopes) Let v, e, and f be positive integers. Prove that there exists a polytope with v vertices, e edges, and f faces if and only if:

$$v - e + f = 2, \qquad v \leq 2f - 4, \text{ and } \qquad f \leq 2v - 4.$$

   You may assume Euler's formula ($v - e + f = 2$); we will prove it later in class.

2. (Polytopes with the same f-vector.) Do there exist two 3-polytopes which are combinatorially different, but have the same number of vertices, edges, and faces?[1]

3. (H and V descriptions of the cube) Prove that our two descriptions of the d-cube C_d coincide:

$$\operatorname{conv}\{+1, -1\}^d = \{\mathbf{x} \in \mathbb{R}^d \mid -1 \leq x_i \leq 1 \text{ for all } 1 \leq i \leq d\}$$

4. (An example of Fourier-Motzkin elimination) Let P be the polygon described by the inequalities:

$$\begin{bmatrix} -1 & -4 \\ -2 & -1 \\ 1 & -2 \\ 1 & 0 \\ 2 & 1 \\ -2 & 6 \\ -6 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} -9 \\ -4 \\ 0 \\ 4 \\ 11 \\ 17 \\ -6 \end{bmatrix}$$

   Use Fourier-Motzkin elimination on P to eliminate the variable x_1. Give the simplest possible description of $\operatorname{proj}_1(P)$.

5. (An "interior" Carathéodory theorem.) Let P be a d-polytope and x be a point in its interior.
   (a) Prove that there exists a subset V of at most $2d$ vertices of P such that x is in the interior of the convex hull of V.
   (b) Prove that the number $2d$ cannot be decreased in general.
   (c) Characterize those P and x for which $2d$ points are needed in V.

6. (Operations on polytopes.)
   (a) If $P, Q \subset \mathbb{R}^d$ are polytopes, prove that $P \cap Q \subset \mathbb{R}^d$ is a polytope.
   (b) If $P, Q \subset \mathbb{R}^d$ are polytopes, prove that $P + Q \subset \mathbb{R}^d$ is a polytope.
   (c) If $P \subset \mathbb{R}^d$ and $Q \subset \mathbb{R}^e$ are polytopes, prove that $P \times Q \subset \mathbb{R}^{d+e}$ is a polytope.

[1](Note. We say P and Q are combinatorially the same polytope if they have the same vertex / edge / face incidence relations. In other words, their vertices can be numbered $p_1, \ldots, p_n$ and $q_1, \ldots, q_n$, respectively, so that $\{p_{a_1}, \ldots, p_{a_k}\}$ is a face of P if and only if $\{q_{a_1}, \ldots, q_{a_k}\}$ is a face of Q for any subset $\{a_1, \ldots, a_k\} \subseteq \{1, \ldots, n\}$.)