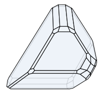
Possibility polytopes 

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Abstract

Sometimes the space of possibilities is a polytope, and that can be very beautiful and very useful! This is the topic of our minicourse.

This is a very rough draft of some lecture notes on this topic. **Please don't share them (yet) outside of this school.**

1 Polytopes

Before we begin: draw three polytopes on your piece of paper, and/or find them in the room around you.

1.1 Vertex and inequality descriptions: the fundamental theorem

Let us begin with some basic definitions. A set $U \subseteq \mathbb{R}^d$ is convex if for every $\mathsf{u}, \mathsf{v} \in U$ the line segment between them is in U. Algebraically:

$$\text{if } u, v \in U \text{ then } \lambda\mathsf{u} + (1-\lambda)\mathsf{v} \in U \text{ for all } 0 \leq \lambda \leq 1$$

The *convex hull* of $V = \{\mathsf{v}_1, \ldots, \mathsf{v}_n\} \subset \mathbb{R}^d$ is the minimal convex set containing V. It is given by:

$$\operatorname{conv}\{\mathsf{v}_1, \ldots, \mathsf{v}_n\} = \left\{ \lambda_1\mathsf{v}_1 + \cdots + \lambda_n\mathsf{v}_n \;:\; \lambda_i \geq 0, \sum_i \lambda_i = 1 \right\}.$$

Can you prove it?

How does this definition match our geometric intuition? For $n = 2$, where is $\lambda\mathsf{u} + (1-\lambda)\mathsf{v}$ located with respect to u and v? For $n = 3$, where is $\lambda\mathsf{u} + \mu\mathsf{v} + \nu\mathsf{w}$ located with respect to $\mathsf{u}, \mathsf{v}, \mathsf{w}$ for $\lambda + \mu + \nu = 1$? What do λ, μ, ν measure?

A *hyperplane* in $\mathbb{R}^d$ is a set of the form $H_{\mathsf{a},b} = \{x \in \mathbb{R}^d \;:\; \mathsf{a} \cdot \mathsf{x} = b\}$ for $\mathsf{a} \in \mathbb{R}^d$ and $b \in \mathbb{R}$. A *halfspace* in $\mathbb{R}^d$ is a set of the form $H_{\mathsf{a},\leq b} = \{x \in \mathbb{R}^d \;:\; \mathsf{a} \cdot \mathsf{x} \leq b\}$.

Theorem 1.1. *(Fundamental Theorem of Polytopes) For a subset $P \subset \mathbb{R}^d$, the following are equivalent*

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1. $P = \operatorname{conv}\{\mathsf{v}_1, \ldots, \mathsf{v}_n\}$ *for some finite set of points* $V = \{\mathsf{v}_1, \ldots, \mathsf{v}_n\}$.

2. P *is bounded and* $P = H_1 \cap \cdots \cap H_m$ *for some finite set of halfspaces* $H = \{H_1, \ldots, H_m\}$.

When this is the case, we call P a polytope. *We write*

$$P = \operatorname{conv}(V) = P(\mathsf{A}, \leq \mathsf{b})$$

where $H_i = H_{\mathsf{a}_i, \leq b_i}$, $\mathsf{A} = [\mathsf{a}_1, \cdots, \mathsf{a}_m]^T$ *and* $\mathsf{b} = [b_1, \cdots, b_m]^T$.

The descriptions above are called the *V-description* and the *inequality description* or *H-description*. When V and H are minimal, we call them the *vertex description* and *facet description*. You might think of the V-description as generative (How do you generate points in the polytope?) and the H-description as restrictive (What must a point satisfy to be in the polytope?), respectively.

The proof of this theorem is beyond the scope of these notes. See [**?**, **?**] for details. As the theorem makes apparent, we will only be interested in convex bounded polytopes. The theory we develop here extends quite naturally to unbounded polyhedra. It does not easily extend to non-convex polyhedral objects.

Remark 1.2. *A (possibly unbounded)* polyhedron *is an intersection of finitely many halfspaces. These objects also have a generative description; for details see [**?**].*

A pentagon.

$$\begin{aligned} \pentagon &= \operatorname{conv}\{((0,0),(2,0),(0,2),(2,1),(1,2)\} \\ &= \{(x,y) \in \mathbb{R}^2 \, : \, 0 \leq x \leq 2, \;\; 0 \leq y \leq 2, \;\; x+y \leq 3\}. \end{aligned}$$

The standard simplex.

$$\begin{aligned} \triangle_{d-1} &= \operatorname{conv}\{e_1, \ldots, e_d\} \\ &= \{x \in \mathbb{R}^d \, : \, x_1 + \cdots + x_d = 1, x_i \geq 0 \text{ for all } i\}. \end{aligned}$$

The standard cube.

$$\begin{aligned} \square_d &= \operatorname{conv}\{\varepsilon_1 e_1 + \cdots + \varepsilon_d e_d \, : \, \varepsilon_i \in \{-1,1\} \text{ for all } i\} \\ &= \{x \in \mathbb{R}^d \, : \, -1 \leq x_i \leq 1 \text{ for all } i\}. \end{aligned}$$

The standard crosspolytope.

$$\begin{aligned} \diamond_d &= \operatorname{conv}\{e_1, -e_1, \ldots, e_d, -e_d\} \\ &= \{x \in \mathbb{R}^d \, : \, \varepsilon_1 x_1 + \cdots + \varepsilon_d x_d \leq 1 \text{ for all } \varepsilon_i \in \{-1,1\}\} \end{aligned}$$

The standard dodecahedron and icosahedron. These may be familiar to you. They will be defined and studied in the Computational Exercise Session.

Question 1.3. *Would you say the polytopes above are combinatorial objects? Why or why not? Discuss this with a friend.*

Note that we are making some non-trivial statements here, that the V-descriptions and H-descriptions above coincide. We should not take for granted that we have both a V-description and an H-description of these polytopes. We should also prove that these descriptions are indeed equal. In general, this is quite hard to do:

Theorem 1.4. *[**?**] Given the H-description of a polyhedron, it is NP-hard to compute its vertices.*

For bounded polytopes this is still open!

Open Problem 1.1. *Given the H-description of a polytope, is it NP-hard to compute its vertices?*

This might not sound hard; why is it? Given the vertices of a polytope, how would you try to compute its inequalities?

Here is a positive statement:

Theorem 1.5. *[**?**] Given the H-description of a polytope with n vertices, m facets, and dimension d, there is an algorithm that can find its vertices in time $O(n^2md)$.*

How do we reconcile the positive and negative statements above?

1.2 Constructions: new polytopes from old

There are several natural ways of combining known polytopes to obtain new polytopes. Let us introduce a few basic constructions.

Intersection. Given two polytopes $P, Q \subset \mathbb{R}^d$, their intersection is

$$P \cap Q = \{r \,:\, r \in P \text{ and } r \in Q\} \subset \mathbb{R}^d.$$

Product. Given two polytopes $P \subset \mathbb{R}^d$ and $Q \subset \mathbb{R}^e$, their product is

$$P \times Q = \{(p, q) \,:\, p \in P, q \in Q\} \subset \mathbb{R}^d \times \mathbb{R}^e \cong \mathbb{R}^{d+e}.$$

Prisms. Given a polytope $P \subset \mathbb{R}^d$, its prism is

$$\mathrm{prism}(P) = P \times [0, 1] \subset \mathbb{R}^d \times \mathbb{R}^1 \cong \mathbb{R}^{d+1}.$$

Pyramid. Given a polytope $P \subset \mathbb{R}^d$, its pyramid is

$$\mathrm{pyr}(P) = \mathrm{conv}(P \times 0, 0 \times 1) \subset \mathbb{R}^d \times \mathbb{R}^1 \cong \mathbb{R}^{d+1}.$$

Minkowski sum. Given two polytopes $P, Q \subset \mathbb{R}^d$, their Minkowski sum is

$$P + Q = \{p + q \,:\, p \in P, q \in Q\} \subset \mathbb{R}^d.$$

Why do these constructions produce polytopes? Try proving it using vertex descriptions and inequality descriptions. You will likely find yourself grateful that both descriptions are available to you.

Later we will see others: Vertex figures, truncations, face lifts.

1.3 Faces

Let $(\mathbb{R}^d)^*$ denote the space of linear functionals on $\mathbb{R}^d$. The standard inner product on $\mathbb{R}^d$ naturally identifies $(\mathbb{R}^d)^*$ with $\mathbb{R}^d$. Still, it is often useful to distinguish these two dual spaces, and draw them separately, next to each other.

A face of a polytope is defined by choosing a direction, and selecting the farthest points of the polytope in that direction. More precisely, let $P \subset \mathbb{R}^d$ be a polytope.and $w \in (\mathbb{R}^d)^*$ be a linear functional. The *w-maximal face of P* is

$$P_w = \{f \in P \,:\, w(f) \geq w(p) \text{ for all } p \in P\}$$

A *face* of a polytope P is any polytope of the form P_w for a linear functional w. By convention, the empty set is also considered a face of P.

The following statements may seem trivial, based on our three-dimensional intuition. We need to be careful, though; our visual intuition can sometimes mislead us when we are dwelling in higher dimensions. Some of the proofs are subtle; see [**?**].

Proposition 1.6. *1. A polytope equals the convex hull of its vertices.*

2. A face of a polytope is a polytope.

3. If G is a face of P and F is a face of G then F is a face of P.

4. If F is a face of P then its vertices are $\mathrm{Vert}(F) = \mathrm{Vert}(P) \cap F$.

5. If F and G are faces of P, then $F \cap G$ is also a face of P.

1.4 A parenthesis: affine spans

An *affine (linear) combination* of $\mathsf{v}_1, \ldots, \mathsf{v}_n$ is a point of the form $\lambda_1\mathsf{v}_1 + \cdots + \lambda_n\mathsf{v}_n$ where $\lambda_1 + \cdots + \lambda_n = 1$. A set $V \subset \mathbb{R}^d$ is *affinely independent* if no point in V is an affine combination of the others. This notion defines a matroid, as introduced in Anastasia Chavez's ECCO course last week.

An *affine subspace* of $\mathbb{R}^d$ can be described equivalently as either

- a subset of $\mathbb{R}^d$ closed under affine combinations,
- the affine span of a finite set $V \subset \mathbb{R}^d$,
- a parallel translate $\mathsf{v} + L$ of a linear subspace L of $\mathbb{R}^d$ for $\mathsf{v} \in \mathbb{R}^d$, or
- the intersection of finitely many hyperplanes.

The *dimension* of an affine subspace $\mathsf{v} + L$ is the dimension of L; it is also the size of the largest affinely independent subset, and the size of the smallest possible generating set.

1.5 Normal fans

A polytope P in $\mathbb{R}^d$ naturally induces a subdivision of $(\mathbb{R}^d)^*$, called the *normal fan.* Each face F of P defines an open and a closed normal cone:

$$\begin{aligned}
\mathcal{N}_P(F)^\circ &= \{w \in (\mathbb{R}^d)^* \,:\, P_w = F\} \\
\mathcal{N}_P(F) &= \{w \in (\mathbb{R}^d)^* \,:\, P_w \geq F\}.
\end{aligned}$$

We have

$$\mathcal{N}_P(F) = \overline{\mathcal{N}_P(F)^\circ} = \bigsqcup_{G \supseteq F} \mathcal{N}_P(G).$$

Notice that there is a bijection between the faces of P and the faces of the normal fan $\mathcal{N}(P)$.

This is a *fan* Σ in the sense that it is a set of full-dimensional cones that intersect properly: for any $C_1, C_2 \in \Sigma$, the intersection $C_1 \cup C_2$ is a face of C_1 and of C_2.

Equivalently, we can think of a fan as a set of cones such that any face of a cone C of Σ is in Σ, and for any two cones $C_1, C_2 \in \Sigma$ the intersection for any $C_1 \cup C_2 \in \Sigma$.

To understand this probably requires doing some examples. Draw the normal fans of the three polytopes that you started the lecture with.

2 Posets: The order of possibilities

In my humble opinion, one of the most frequent mistakes made by mathematicians is to force things to be on a line, when they don't want to be. Non-mathematicians do this too, trying to decide who/what is the "best" person or team or album or rank them in linear order, when there is simply no good way to do that. when there is simply not a best one. Most sets we encounter do not naturally have a linear order, and forcing them to have one – for example by labeling its elements $1, 2, \ldots n$ – can be misleading, confusing, obscuring. By contrast, many sets we encounter do naturally have a partial order – where some pairs of elements are comparable and some are not – and that structure is often very useful.

2.1 Posets

A **partially ordered set** or **poset** $(P, \leq)$ is a set P together with a binary relation $\leq$, called a *partial order*, such that

- For all $p \in P$, we have $p \leq p$.
- For all $p, q \in P$, if $p \leq q$ and $q \leq p$ then $p = q$.
- For all $p, q, r \in P$, if $p \leq q$ and $q \leq r$ then $p \leq r$.

We say that $p < q$ if $p \leq q$ and $p \neq q$. We say that p and q are **comparable** if $p < q$ or $p > q$, and they are **incomparable** otherwise. We say that q **covers** p if $q > p$ and there is no $r \in P$ such that $q > r > p$. When q covers p we write $q \gtrdot p$.

Example 2.1. *Posets are abundant in mathematics and beyond. Let us discuss just a few examples.*

1. (Chain) *The poset* $\mathbf{n} = \{1, 2, \ldots, n\}$ *with the usual total order.*
2. (Boolean lattice) *The poset* 2^A *of subsets of a set* A*, where* $S \leq T$ *if* $S \subseteq T$*.*
3. (Divisor lattice) *The poset* D_n *of divisors of* n*, where* $c \leq d$ *if* c *divides* d*.*
4. (Partition lattice) *The poset* Π_n *of set partitions of* $[n]$*, where* $\pi \leq \rho$ *if* π refines ρ*;* i.e.*, every block of* ρ *is a union of blocks of* π*.*
5. (Face poset of a polytope) *The poset* $L(P)$ *of faces of a polytope* P*, where* $F \leq G$ *if* F *is a face of* G*.*

Think of a few more examples, coming from areas of mathematics (or beyond) that you are interested in.

The **Hasse diagram** of a finite poset P is obtained by drawing a dot for each element of P and an edge going down from p to q if p covers q. Figure 1 below shows the Hasse diagrams of some of the posets above. In particular, the Hasse diagram of $2^{[n]}$ is the 1-skeleton of the n-dimensional cube.

A poset is *ranked* if there is a function $r : P \to \mathbb{Z}$ such that minimum elements have rank 0, and $p < q$ implies $r(q) - r(p) = 1$. Equivalently, every maximal chain from x to y has the same length.

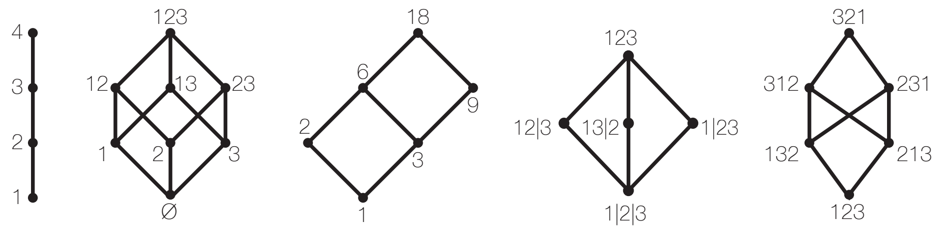

Figure 1: The Hasse diagrams of the chain $\mathbf{4}$, Boolean lattice $2^{[3]}$, divisor lattice D_{18}, partition lattice Π_3, Bruhat order S_3, and polytopal face poset $L(\pentagon)$ (missing).

Proposition 2.2. *All the posets of Example 2.1 are ranked.*

What are the rank functions?

The **direct product** $P \times Q$ is the poset on $P \times Q$ where $(p, q) \leq (p', q')$ if $p \leq p'$ and $q \leq q'$.We have already seen examples of product posets. The Boolean lattice is $2^A \cong \mathbf{2} \times \cdots \times \mathbf{2}$. Also, if $n = p_1^{t_1} \cdots p_k^{t_k}$ is the prime factorization of n, then $D_n \cong (\mathbf{t_1} + \mathbf{1}) \times \cdots \times (\mathbf{t_k} + \mathbf{1})$. Prove this!

2.2 Lattices

A poset is a **lattice** if every two elements p and q have a least upper bound $p \vee q$ and a greatest lower bound $p \wedge q$, called their **meet** and **join**, respectively. This additional algebraic structure can be very beneficial.

To prove that a finite poset P is a lattice, it is sufficient to check that it has a maximum element $\widehat{1}$ and any $x, y \in P$ have a meet $x \wedge y$; then the join of x and y will be the (necessarily non-empty) meet of their common upper bounds. Similarly, it suffices to check that P has a minimum element $\widehat{0}$ and any $x, y \in P$ have a join $x \vee y$.

Proposition 2.3. *The families of Example 2.1 are lattices:*

1. *Every chain is a lattice.*
2. *Every Boolean poset is a lattice.*
3. *Every divisor poset is a lattice.*
4. *Every partition poset is a lattice.*
5. *Every polytopal face poset is a lattice.*

We will omit the proof in these notes, but we invite you to describe the meet and/or the join for each of the families above.

2.3 Intervals

For $p \leq q$ we define the interval

$$[p, q] = \{r \in P \,:\, p \leq r \leq q\}.$$

We let $\mathrm{Int}(P)$ be the set of intervals of P. Also let $P_{\geq p} = \{q \in P \,:\, q \geq p\}$.

Proposition 2.4. *The families of Example 2.1 behave well under taking intervals:*

1. *Every interval of a chain is a chain.*
2. *Every interval of a Boolean lattice is a Boolean lattice.*
3. *Every interval of a divisor poset is a divisor poset.*
4. *Every interval of a partition poset is a product of partition posets.*
5. *Every interval of a polytopal face poset is a polytopal face poset.*

We invite you to provide the proofs for 1.–4. Let us prove 5., which is most relevant to these notes. To prove it we need a new construction.

Definition 2.5. *Let P be a polytope and v be a vertex. The* vertex figure P/v *of P at v is a generic slice of P very close to v. More concretely, suppose $\mathsf{v} = P_{\mathsf{w}}$ with $\mathsf{w} \cdot \mathsf{v} = c$. Then for $c' = c - \epsilon$ we let*

$$P/\mathsf{v} = P \cup \{x \,:\, \mathsf{w} \cdot \mathsf{x} = c'\}$$

Notice that the faces of the vertex figure P/v are in bijection with the faces of P containing v.

Proof of Proposition 2.4.5. Let P be a polytope. It suffices to prove the statement for a maximal non-trivial interval of $L(P)$; the result will then follow by induction. There are two kinds of maximal intervals.

1. Let P be a polytope and consider a maximum element y of $L(P) - \widehat{1}$. Then y corresponds to a facet F of P, and the elements less than y in $L(P)$ are the faces of P contained in F. Therefore $[\widehat{0}, y] \cong L(F)$.

2. Now consider a maximum element x of $L(P) - \widehat{0}$. Then x corresponds to a vertex v of P, and the elements greater than y in $L(P)$ are the faces of P containing v, which are in bijection with the faces of P/v. Therefore $[x, \widehat{1}] \cong L(P/\mathsf{v})$. □

2.4 Möbius functions

Consider a finite poset P with a minimum element $\widehat{0}$.

Definition 2.6. *The (one-variable) Möbius function $\mu : P \to \mathbb{Z}$ is defined by*

$$\mu(p) = \begin{cases} 1 & \text{if } p = \widehat{0}, \\ -\sum\limits_{r<p} \mu(r) & \text{otherwise.} \end{cases} \tag{1}$$

Equivalently, the sum of the Möbius numbers of any non-trivial lower interval is 0.

What are the Möbius functions of the posets in Figure **??**?

Let's compute it for L(square pyramid) together. What do you notice?

Computing Möbius functions is a very important problem, because **the Möbius function is the poset analog of a derivative**; and as such, it is a fundamental invariant of a poset. This problem is fun, beautiful, and interesting, but that is a story for another day. You can read more about it in [**?**, Section 4.5].

Definition 2.7. *The (two-variable) Möbius function $\mu : Int(P) \to \mathbb{Z}$ is defined by $\mu(p,q) = \mu_{\geq p}(q)$, where $\mu_{\geq p} = \mu_{P_{\geq p}}$ denotes the Möbius function of the poset $P_{\geq p}$. Equivalently,*

$$\mu(p,q) = \begin{cases} 1 & \text{if } p = q, \\ -\sum\limits_{p \leq r < q} \mu(p,r) & \text{otherwise.} \end{cases} \tag{2}$$

Equivalently, the sum of the numbers $\mu(p,-)$ in any non-trivial interval $[p,q]$ is 0.

It turns out that the sum of the numbers $\mu(-,q)$ in any non-trivial interval $[p,q]$ is also 0.

2.5 Polytopes and dimensions

The *dimension* of a polytope is the dimension of its affine span. Faces of dimension 0 and 1 are called *vertices* and *edges*. Faces of codimension 1 and 2 are called *facets* and *ridges*.

The *f-vector* of P is $f(P) = (f_{-1}, f_0, f_1, \ldots, f_d)$, where f_i is the number of i-dimensional faces. The *f-polynomial* is $f_P(x) = \sum_{i=0}^{d} f_i x^i$. In the Exercise Session you will compute these for the polytopes we have discussed so far.

Proposition 2.8. *For the face lattice of a polytope, $\mu(F) = (-1)^{\dim F}$.*

The key property here is the familiar formula due to Euler in 1752:

$$V - E + F = 2.$$

Klee called this the "*first landmark in the theory of polytopes*". Alexandroff and Hopf called it the "*first important event in topology*". More generally, we have:

Theorem 2.9. *For any d-polytope.*

$$-f_{-1} + f_0 - f_1 + f_2 - \cdots + (-1)^d f_d = 0$$

.

Definition 2.10. *A poset P is* Eulerian *if it is ranked and*

$$\mu(x,y) = (-1)^{r(y)-r(x)}$$

Theorem 2.11. *Face posets of polytopes are Eulerian:*

$$\mu_{L(P)}(F,G) = (-1)^{\dim G - \dim F} \text{ for any faces } F \subseteq G.$$

Proof. First let us prove that $\mu(G) = (-1)^{\dim G}$ by induction on $\dim G$. For $\dim G = -1$ we have $\mu(\widehat{0}) = 1$ by definition. Assuming that we have proved it up to dimension $k-1$, consider a face G of dimension k. For any $F \subsetneq G$ we have $\mu(F) = (-1)^{\dim F}$, and then

$$\begin{aligned} 0 &= \sum_{F \leq G} \mu(F) \\ &= \sum_{F < G} (-1)^{\dim(F)} + \mu(G) \\ &= -f_{-1}(G) + f_0(G) - f_1(G) + \cdots \pm f_{d-1}(G) \mp \mu(G) \end{aligned}$$

and Euler tells us that the $\mu(G)$ that makes this equation work is $\mu(G) = (-1)^{\dim G}$.

To prove it in general, note that upper intervals of the face poset $L(P)$ are face posets of iterated face figures $((P/v)/v')\ldots$, so their Möbius numbers are also alternating 1 and -1s. □

2.6 Möbius inversion formula

There is a very beautiful story that I will not get to say much about, but I would like to at least mention it briefly.

In enumerative combinatorics, there are many situations where have a set U of objects, and a natural way of assigning to each object u of U an element $f(u)$ of a poset P. We are interested in counting the objects in U that map to a particular element $p \in P$. Often we find that it is much easier to count the objects in s that map to an element **less than or equal to**[1] p in P. The following theorem tells us that this easier enumeration is sufficient for our purposes, as long as we can compute the Möbius function of P.

Theorem 2.12. (Möbius Inversion formula) *Let P be a poset and let $f, g : P \to \mathbb{F}$ be functions from P to a field $\mathbb{F}$. Then*

1. $g(p) = \sum_{q \geq p} f(q)$ *for all* $p \in P \iff f(p) = \sum_{q \geq p} \mu(p,q) g(q)$ *for all* $p \in P$, *and*
2. $g(p) = \sum_{q \leq p} f(q)$ *for all* $p \in P \iff f(p) = \sum_{q \leq p} \mu(q,p) g(q)$ *for all* $p \in P$.

This theorem simultaneously generalizes:
• discrete derivatives, when $P = [n]$
• the Möbius inversion formula of number theory, when $P = D_n$
• the inclusion-exclusion formula, when $P = 2^A$

[1] or greater than or equal to

• the Ehrhart reciprocity, when P is the face lattice of a polytope.

In his paper [**?**], which pioneered the use of the Möbius inversion formula as a tool for counting in combinatorics, Rota described this enumerative philosophy as follows:

> It often happens that a set of objects to be counted possesses a natural ordering, in general only a partial order. It may be unnatural to fit the enumeration of such a set into a linear order such as the integers: instead, it turns out in a great many cases, that a more effective technique is to work with the natural order of the set. One is led in this way to set up a "difference calculus" relative to an arbitrary partially ordered set.

Indeed, one may think of the Möbius function as a poset-theoretic analog of the Fundamental Theorem of Calculus: g is analogous to the integral of f, as it stores the cumulative value of this function. Like the Fundamental Theorem of Calculus, the Möbius inversion formula tells us how to recover the function f from its cumulative values.

It is not difficult to prove Theorem 2.12 directly, but it takes some practice to truly appreciate it and make it part of your toolkit. Again, this will be a story for another time.

3 Combinatorial polytopes: motivating examples

The following is the main question of this minicourse.

3.1 Which Eulerian lattices are polytopal?

Question 3.1. *Suppose a family $\mathcal{F}$ of combinatorial objects has the structure of a poset $L(\mathcal{F})$. Is there a polytope P* realizing *$\mathcal{F}$ in the sense that*

$$L(P) \cong L(\mathcal{F})?$$

A necessary condition is that $L(P)$ should be an Eulerian lattice. In the exercises you will see that this is not sufficient.

At this moment in mathematical history, answering this question is more of an art than a science. I designed this minicourse with the goal of beginning to collect the state of the art on this problem.

Let's begin with two important motivating examples.

3.2 Subsets

Consider the poset 2^E of subsets of E.

Theorem 3.2. *The standard simplex Δ_E realizes the Boolean poset 2^E.*

3.3 Signed subsets

Consider the poset $\{-,0,+\}^E$ of signed subsets of E.

Theorem 3.3. *The cross-polytope* $\Diamond_E$ *realizes the poset* $\{-,0,+\}^E$

3.4 Permutations

Consider the poset $O\Pi_n$ of ordered set partitions of $[n]$, where $A \leq B$ if A *refines* B; *i.e.*, every block of A is a union of consecutive blocks of B.

Theorem 3.4. *The* permutahedron*, defined as*

$$\begin{aligned} \mathrm{Perm}_{n-1} &= \mathrm{conv}(\pi_1, \ldots, \pi_n) \,:\, \pi \textit{ is a permutation of } [n]) \\ &= \left\{ x \in \mathbb{R}^n \,:\, \sum_{i \in [n]} x_i = \frac{1}{2} n(n+1), \sum_{i \in I} x_i \geq \frac{1}{2} i(i+1), \textit{ for all } I \subseteq [n] \right\} \end{aligned}$$

realizes the poset $O\Pi_n$ *of ordered set partitions of* $[n]$.

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Consider the poset $O\Pi_n$ of ordered set partitions of $[n]$, where $A \leq B$ if A *refines* B; *i.e.*, every block of A is a union of consecutive blocks of B.

Theorem 3.5. *The* permutahedron*, defined as*

$$\begin{aligned} \mathrm{Perm}_{n-1} &= \mathrm{conv}(\pi_1, \ldots, \pi_n) \,:\, \pi \textit{ is a permutation of } [n]) \\ &= \left\{ x \in \mathbb{R}^n \,:\, \sum_{i \in [n]} x_i = \frac{1}{2} n(n+1), \sum_{i \in I} x_i \geq \frac{1}{2} i(i+1), \textit{ for all } I \subseteq [n] \right\} \end{aligned}$$

realizes the poset $O(\Pi)_n$ *of ordered set partitions of* $[n]$.

3.6 Triangulations

Consider the poset A_n of subdivisions of an n-gon, where $S \leq T$ if S *refines* T; *i.e.*, every polygon in S is a union of polygons in T.

Theorem 3.6. *The* associahedron*, defined as*

$$\mathrm{Assoc}_{n-1} = \left\{ x \in \mathbb{R}^n \,:\, \sum_{i \in [n]} x_i = \frac{1}{2} n(n+1), \sum_{i \in I} x_i \geq \frac{1}{2} i(i+1), \textit{ for all intervals } I \subseteq [n] \right\}$$

realizes the poset $Subdiv_{n+2}$ *of subdivisions of the* $(n+2)$*-gon.*

The formula for the vertices is very pretty, and is best explained in a picture. I will draw it in class.

4 Combinatorial polytopes: how to construct them

4.1 A strategy

A d-polytope is *simple* if every vertex is on d facets. A d-polytope is *simplicial* if every proper face(t) is a simplex.

Strategy 4.1. *To prove that a poset $L(\mathcal{F})$ of combinatorial objects $\mathcal{F}$ has a polytope realizing it:*

1. *Find a fan Σ realizing $\mathcal{F}$*
2. *Prove that $L(\Sigma)^{op} \cong L(\mathcal{F})$*
3. *Find a polytope P whose normal fan is Σ.*
4. *Prove that $\mathcal{N}(P) = \Sigma$.*

Step 1 requires creativity, often relying on earlier known examples for inspiration. Step 2 requires understanding the linear algebra / linear inequalities of Σ, and proving that its combinatorial structure matches the combinatorics of $\mathcal{F}$ Step 3 often requires creativity as well. Step 4 can be done by hand, but we will see a useful general tool.

Definition 4.2. (Wall-crossing inequalities) *Let Σ be a simple fan with set of rays $\Sigma(1)$. Let σ and σ' be two neighboring facets, separated by a $(d-1)$-dimensional wall τ. Consider the $d-1$ rays $\rho_1, \ldots, \rho_{d-1}$ generating τ and the two generating rays ρ, ρ' of σ, σ', respectively, that are not in τ. Up to scaling, there is a unique linear dependence of the form*

$$c \cdot \rho + c' \cdot \rho' = \sum_{i=1}^{d-1} c_i \cdot \rho_i \tag{3}$$

with $c, c' > 0$. To the wall τ we associate the wall-crossing inequality

$$I_{\Sigma,\tau}(h) := c \cdot h(\rho) + c' \cdot h(\rho') - \sum_{i=1}^{d-1} c_i \cdot h(\rho_i) \geq 0. \tag{4}$$

Theorem 4.3. (Wall-Crossing Criterion) *[**?**, Theorems 6.1.5–6.1.7] For $h : \Sigma(1) \to \mathbb{R}$ consider the polytope*

$$P(h) = \{x \in \mathbb{R}^d \,:\, \rho \cdot x \leq h(\rho) \text{ for } \rho \in \Sigma(1)\}.$$

The normal fan of $P(h)$ equals Σ if and only h satisfies the wall-crossing inequalities $I_{\Sigma,\tau}(h) > 0$, as defined in (4), for each wall τ of Σ.