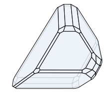
Possibility Polytopes 

Federico Ardila–Mantilla*

These are the exercises of Federico Ardila–Mantilla's course at ECCO Tico 2026. These are best done in group discussions full of curiosity, thoughtful communication, patience, and generosity.

You may proceed in any order you like, but I do recommend doing at least one computational and one conceptual exercise. Problems are meant to pique your interest, and you are welcome to think about them and discuss them with your friends – but probably that is best left for after the exercise session. **Exercises in bold** are recommended.

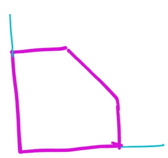
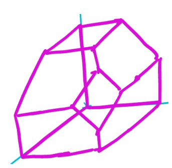
1 Exercise Session 1.

1.1 Computational exercises

1. (H and V descriptions) **Find the V and H description of**

   (a) a polygon in $\mathbb{R}^2$ as shown with facet directons $-e_1, -e_2, e_1, e_2, e_1 + e_2$.

   (b) a 3-polytope in $\mathbb{R}^3$ as shown with facet directons $-e_1, -e_2, -e_3, e_1, e_2, e_3, e_1 + e_2, e_1 + e_3, e_1 + e_2 + e_3$.

2. (Normal fans) **Sketch the normal fans of ⬠, $\square_3$, $\triangle_3$, $\lozenge_3$.**

3. (Cubes) If you have never done this before, draw the hypercube $\square_4$. If you have time tonight, draw or build the hypercube $\square_6$.

*Queen Mary University of London; San Francisco State University. f.ardila@qmul.ac.uk

1.2 Conceptual exercises.

4. (H and V descriptions)

   (a) **Prove that our two descriptions of the d-cube $\square_d$ coincide**:

   $$\mathrm{conv}\{+1,-1\}^d = \{\mathbf{x} \in \mathbb{R}^d \mid -1 \leq x_i \leq 1 \text{ for all } 1 \leq i \leq d\}$$

   (b) Prove that our two descriptions of the d-crosspolytope $\lozenge_d$ coincide:

   $$\mathrm{conv}\{\mathbf{e}_1, -\mathbf{e}_1, \dots, \mathbf{e}_d, -\mathbf{e}_d\} = \{\mathbf{x} \in \mathbb{R}^d \mid \mathbf{a}\,\mathbf{x} \leq 1 \text{ for all } \mathbf{a} \in \{+1,-1\}^d \subset (\mathbb{R}^d)^*\}$$

5. (Operations on polytopes.)

   (a) If $P, Q \subset \mathbb{R}^d$ are polytopes, prove that $P \cap Q \subset \mathbb{R}^d$ is a polytope.

   (b) If $P, Q \subset \mathbb{R}^d$ are polytopes, prove that $P + Q \subset \mathbb{R}^d$ is a polytope.

   (c) If $P \subset \mathbb{R}^d$ and $Q \subset \mathbb{R}^e$ are polytopes, prove that $P \times Q \subset \mathbb{R}^{d+e}$ is a polytope.

   Think about how you might obtain their V and H descriptions from the original polytopes.

1.3 Problems

6. (Matroid polytopes) Prove that the following two descriptions of the matroid polytope of a matroid M coincide:

   $$\mathrm{conv}\left\{\sum_{b \in B} \mathbf{e}_b \,:\, B \text{ basis}\right\} = \left\{\mathbf{x} \in \mathbb{R}^d \mid \sum_{i=1}^{d} x_i = r(M), \sum_{a \in A} x_s \leq r(A) \text{ for all } \emptyset \subsetneq A \subsetneq [d]\right\}$$

   What are the vertices of this polytope? What are the facets?

7. (A linear programming example.) An aid agency receives applications from n volunteers, and seeks to assign some of them to $m < n$ positions. Each location gets one volunteer. They are shown descriptions of the applicants, and answer the question: on a scale of 1-5, how satisfied would you be with this applicant? The goal of the agency is to assign volunteers to locations in a way that maximizes the total satisfaction (the sum of the individual satisfaction scores) of the agencies.

   Phrase this as a polytopal question. Find the V-description and the H-description of the relevant polytope. What can you say about this polytope?

2 Exercise Session 2.

These are the exercises of Federico Ardila–Mantilla's course at ECCO Tico 2026. These are best done in group discussions full of curiosity, thoughtful communication, patience, and generosity.

You may proceed in any order you like, but I do recommend doing at least one computational and one conceptual exercise. Problems are meant to pique your interest, and you are welcome to think about them and discuss them with your friends – but probably that is best left for after the exercise session. **Exercises in bold** are recommended.

2.1 Computational exercises.

1. (Computing posets: examples)
   (a) Compute the divisor poset D_{60}.
   (b) Compute the partition poset Part_4.
   (c) **Compute the face poset of** $\square_3$.

2. (Computing Möbius functions: examples)
   (a) Compute the Möbius function of the divisor poset D_{60}.
   (b) Compute the Möbius function of the partition poset Part_4.
   (c) **Compute the Möbius function of the face poset of** $\square_3$**..**

2.2 Conceptual exercises.

3. (Poset factorization)
   (a) Prove that the Boolean lattice is $2^A \cong \mathbf{2} \times \cdots \times \mathbf{2}$.
   (b) Prove that $D_n \cong (\mathbf{t_1+1}) \times \cdots \times (\mathbf{t_k+1})$ where $n = p_1^{t_1} \cdots p_k^{t_k}$ is the prime factorization of n.

4. (Intervals) Prove that
   (a) Every interval of a chain is a chain.
   (b) **Every interval of a Boolean poset is a Boolean poset**.
   (c) Every interval of a divisor poset is a divisor poset.
   (d) Every interval of a partition poset is a product of partition posets.
   (e) **Every interval of a polytopal face poset is a polytopal face poset.**

5. (Lattices) Prove that
   (a) Every chain is a lattice.
   (b) **Every Boolean poset is a lattice.**
   (c) Every divisor poset is a lattice.
   (d) Every partition poset is a lattice.

(e) **Every polytopal face poset is a lattice.**

6. (f-polynomials)

   (a) Find the f-polynomial of $\pentagon$, Δ_{d-1}, $\square_d$, $\lozenge_d$, the dodecahedron and the icosahedron.

   (b) Find the f-polynomial of the pyramid $\mathrm{pyr}(P)$ over a polytope P in terms of the f-polynomial of P.

   (c) Find the f-polynomial of the product $P \times Q$ of two polytopes P and Q in terms of the f-polynomials of P and Q.

2.3 Problems.

7. (Computing Möbius functions: families)

   (a) Compute the Möbius function of the divisor poset D_n.

   (b) Compute the Möbius function of the face posets of $\pentagon, \triangle_n, \square_n, \lozenge_n$.

   (c) (*) Compute the Möbius function of the partition poset Part_n.

8. (Ehrhart theory) For a polytope P with integer vertices, let $E_P(n)$ be the number of lattice points in the nth dilate nP, and $E_P^o(n)$ be the number of interior lattice points in nP.

   (a) Express the Ehrhart function of P in terms of the interior Ehrhart function of its faces.

   (b) Express the interior Ehrhart function of P in terms of the Ehrhart function of its faces.

   (c) Prove that $|E_P^o(n)| = |E_P(-n)|$.

   (d) Prove that these functions are polynomial in n.

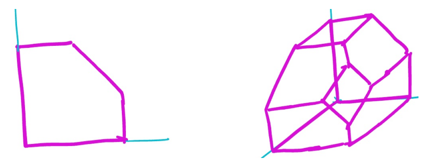
   The function $E_P(n)$ is called the *Ehrhart polynomial* of P.

3 Exercise Sessions 3 and 4.

3.1 Computational exercises.

1. (Non-polytopal Eulerian examples)
   (a) Find an Eulerian poset that is not a polytopal face poset.
   (b) Find an Eulerian lattice that is not a polytopal face poset.

2. (Computing type cones: examples)
   (a) Find **all** polygons in $\mathbb{R}^2$ as shown with facet directons $-e_1, -e_2, e_1, e_2, e_1+e_2$.
   (b) Find **all** 3-polytopes in $\mathbb{R}^3$ as shown with facet directons $-e_1, -e_2, -e_3, e_1, e_2, e_3, e_1 + e_2, e_1+e_3, e_1+e_2+e_3$.
   (c) Find **all** polytopes in $\mathbb{R}^d$ with the same normal fan as the cube $\square_d$.



3. (Eulerian lattices in nature) Compute the posets and verify that they are Eulerian lattices.
   (a) The poset $O\Pi_4$ of ordered set partitions of $[4]$.
   (b) The poset Subdiv_6 of subdivisions of a hexagon.

3.2 Conceptual exercises.

4. (The type cone of the braid fan) Prove that the type cone of the braid fan in $\mathbb{R}^E$ is the submodular cone

$$\{f : 2^E \to \mathbb{R} \,:\, f(A\cup B) + f(A\cap B) \leq f(A) + f(B)\}$$

5. (The type cone of the associahedral fan) Compute the type cone of the associahedral fan. Prove that it is simplicial.

6. (Minkowski sum decompositions)
   (a) Prove that

$$\mathrm{Perm}_n = \sum_{1\leq i<j\leq n} \mathrm{conv}(e_i, e_j)$$

   (b) Prove that

$$\mathrm{Assoc}_n = \sum_{1\leq i<j\leq n} \mathrm{conv}(e_i, e_{i+1}, \ldots, e_j)$$

3.3 Problems.

7. (Vertices of the associahedron) Find and prove a formula for the vertex v_T of the associahedron Assoc_{n+2} corresponding to a triangulation (or equivalently a binary tree) T.

8. (The cyclic polytope is very neighborly.) The cyclic polytope is a d-polytope in $\mathbb{R}^d$ with n vertices defined as follows:
$$C(n,d) = \operatorname{conv} \mathbf{x}(t_1), \ldots, \mathbf{x}(t_n)$$
where $\mathbf{x}(t) = (t, t^2, \ldots, t^d)$.

   (a) *Gale's evenness condition* describes the facets of the cyclic polytope: A d-subset $I \subseteq [n]$ defines a facet if and only if any two elements of $[n] - I$ are separated by an even number of elements of I. Prove this.

   (b) Conclude that the cyclic polytope $C_d(n)$ is $\left(\lfloor \frac{d}{2} \rfloor\right)$ - neighborly.

9. (How neighborly can you be?) If a d-dimensional polytope is $\left(\lfloor \frac{d}{2} \rfloor + 1\right)$ - neighborly, prove that it must be a simplex.

(Recall that a polytope is k-neighborly if any set of k vertices forms a face.)