Recall the

$\boxed{\text{Malvenuto-Reutenauer Hopf algebra of permutations}}$

$$\boxed{\mathfrak{S}Sym = \mathbb{K}\{\mathcal{F}_u : u \in S_n \text{ for some } n \geq 0\}}$$

product: shuffling

$$\mathcal{F}_{12} \cdot \mathcal{F}_{312} = \mathcal{F}_{12534} + \mathcal{F}_{15234} + \mathcal{F}_{15324} + \mathcal{F}_{15342} + \mathcal{F}_{51234}$$
$$+ \mathcal{F}_{51324} + \mathcal{F}_{51342} + \mathcal{F}_{53124} + \mathcal{F}_{53142} + \mathcal{F}_{53412}$$

Coproduct: cutting

$$\Delta(\mathcal{F}_{42531}) = 1 \otimes \mathcal{F}_{42531} + \mathcal{F}_1 \otimes \mathcal{F}_{2431} + \mathcal{F}_{21} \otimes \mathcal{F}_{321}$$
$$+ \mathcal{F}_{213} \otimes \mathcal{F}_{21} + \mathcal{F}_{3142} \otimes \mathcal{F}_1 + \mathcal{F}_{42531} \otimes 1$$

How are all of these related?

Fact

- As an algebra, $\mathfrak{S}Sym$ is free
- The antipode has infinite order.

A Sym, NSym, NCSym

In class someone suggested a map

$$Sym \longrightarrow NCSym$$
$$m_\lambda(x) \longmapsto m_\lambda(y)$$

(x comm, y non-comm)

This is not an algebra map:

$$f(m_1^3) = f((\textstyle\sum x_i)^3)$$
$$= f(\textstyle\sum x_i^3 + 3\sum x_i^2 x_j + 6\sum x_i x_j x_k)$$
$$= \textstyle\sum y_i^3 + 3\sum y_i^2 y_j + 6\sum y_i y_j y_k$$

$$f(m_1)^3 = (\textstyle\sum y_i)^3$$
$$= \textstyle\sum y_i^3 + \sum y_i^2 y_j + \sum y_i y_j y_i + \sum y_i y_j^2 + \cdots$$

We do have a map

$$\boxed{\begin{array}{c} \chi : NCSym \twoheadrightarrow Sym \\ (y_i \longmapsto x_i) \end{array}}$$

$$\chi\Big(\sum_{i \neq j} x_i x_j x_i\Big) = \sum_{i \neq j} x_i^2 x_j$$

which forgets that the variables are non-comm.

We also have a map

$$\boxed{\begin{array}{c} \psi : NSym \twoheadrightarrow Sym^* \\ h_i \longmapsto h_i \end{array}}$$

$$\chi(h_2 h_3 h_2) = h_{322}$$

which forgets that the h_i's are non-comm

These are clearly surjective, so

Sym is a quotient of NSym, NCSym

NSym and NCSym are also related.

The map

$$\chi : NCSym \twoheadrightarrow Sym$$

has a "pullback"

$$\boxed{\begin{aligned} \tilde{\chi} : Sym &\longrightarrow NCSym \\ m_\lambda &\longmapsto \sum_{\lambda(\pi)=\lambda} m_\pi \end{aligned}}$$

$$\tilde{\chi}\Big(\sum_{i \neq j} x_i^2 x_j\Big) = \sum_{i\neq j} x_i^2 x_j + \sum_{i \neq j} x_i x_j x_i + \sum_{i\neq j} x_i x_j^2$$

such that

$$\chi \circ \tilde{\chi} = \text{identity}$$

This gives

$$\boxed{\begin{aligned} I : NSym &\hookrightarrow NCSym \\ h_n &\longmapsto \tilde{\chi}(h_n) \end{aligned}}$$

$$\begin{tikzcd} NSym \arrow[rr, "I"] \arrow[dr, "\psi"'] & & NCSym \\ & Sym \arrow[ur, "\tilde{\chi}"'] & \end{tikzcd}$$

Then the following diagram commutes:

$$\begin{tikzcd} NSym \arrow[r, hook, "I"] \arrow[d, "\psi"'] & NCSym \arrow[d, "\chi"] \\ Sym \arrow[ur, dotted, "\tilde{\chi}"] \arrow[r, equal] & Sym^* \end{tikzcd}$$

(B) Sym, NSym, QSym, 𝔖Sym

Aside from $\begin{tikzcd} NSym \arrow[d] \\ Sym \end{tikzcd}$, I have a natural inclusion

$$\boxed{Sym \hookrightarrow QSym}$$

Let's bring 𝔖Sym into the picture.

Given a permutation $u \in S_n$, let the <u>descent set</u> of u be

$$\boxed{Des(u) = \{ i \in [n-1] : u_i > u_{i+1} \}}$$

For example

$$Des(\underline{7}23\underline{6}1\underline{5}4) = \{1,4,6\}$$

which gives the composition 1321 of 7

Prop The map
$$\begin{aligned} D : \mathfrak{S}Sym &\twoheadrightarrow QSym \\ F_u &\longmapsto F_{c(Des(u))} \end{aligned}$$
is a morphism of Hopf algebras (Malvenuto)

This is surjective since the F_S are a basis for QSym.

Another nice fact:

> **Thm** $\mathfrak{S}Sym$ is self-dual as a graded Hopf algebra

If $\{F_\sigma^*\}$ is the basis of $\mathfrak{S}Sym^*$ dual to $\{F_\sigma\}$ then an isomorphism is

$$\begin{aligned} \mathfrak{S}Sym^* &\longrightarrow \mathfrak{S}Sym \\ F_\sigma^* &\longmapsto F_{\sigma^{-1}} \end{aligned}$$

Then

$$\mathfrak{S}Sym \longrightarrow QSym$$

gives a dual map

$$\begin{tikzcd} NSym \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & \mathfrak{S}Sym \arrow[d, phantom, "\cong" rotate=90] \\ QSym^* & \mathfrak{S}Sym^* \end{tikzcd}$$

$$\left(F_I \longmapsto \sum_{Des(\sigma)=I} F_\sigma \right)$$

and we have a commutative diagram.

$$\boxed{\begin{tikzcd} NSym \arrow[r, hook] \arrow[d] & \mathfrak{S}Sym \arrow[d] \\ Sym \arrow[r, hook] & QSym \end{tikzcd}}$$

In fact, these two diagrams fit into a larger diagram:

$$\begin{tikzcd} NSym \arrow[rr] \arrow[dd] \arrow[dr] & & \mathfrak{S}Sym \arrow[dd] \arrow[dr, dashed] & \\ & NCSym \arrow[rr, dashed] \arrow[dd] & & NCQSym \arrow[dd, dashed] \\ Sym \arrow[rr] \arrow[dr] & & QSym \arrow[dr, dashed] & \\ & Sym \arrow[rr, dashed] & & QSym \end{tikzcd}$$

There is so much more to say about this! (Much of it open)