

Recall the

Malvenuto-Proust-Hopf algebra of permutations

$$\mathcal{G}\text{Sym} = \mathbb{K}\{\mathcal{F}_\nu : \nu \in S_n \text{ for some } n \geq 0\}$$

Product: shuffling

$$\mathcal{F}_{12} \cdot \mathcal{F}_{312} = \mathcal{F}_{12534} + \mathcal{F}_{15234} + \mathcal{F}_{15324} + \mathcal{F}_{15342} + \mathcal{F}_{51234} \\ + \mathcal{F}_{51324} + \mathcal{F}_{51342} + \mathcal{F}_{53124} + \mathcal{F}_{53142} + \mathcal{F}_{53412}$$

Coproduct: cutting

$$\Delta(\mathcal{F}_{42531}) = 1 \otimes \mathcal{F}_{42531} + \mathcal{F}_1 \otimes \mathcal{F}_{2431} + \mathcal{F}_{21} \otimes \mathcal{F}_{321} \\ + \mathcal{F}_{213} \otimes \mathcal{F}_{21} + \mathcal{F}_{3142} \otimes \mathcal{F}_1 + \mathcal{F}_{42531} \otimes 1$$

How are all of these related?

Fact

- As an algebra,  $\mathcal{G}\text{Sym}$  is free
- The antipode has infinite order.

### (A) $\text{Sym}, \text{NSym}, \text{NCSym}$

In class someone suggested a map

$$\text{Sym} \rightarrow \text{NCSym}$$

$$m_2(x) \mapsto m_2(y) \quad (x \text{ comm, } y \text{ non-comm})$$

This is not an algebra map:

$$f(m_2^3) = f(\sum x_i y_i^2) \\ = f(\sum x_i^3 + 3 \sum x_i^2 y_i + 6 \sum x_i y_i x_i) \\ = \sum x_i^3 + 3 \sum x_i^2 y_i + 6 \sum x_i y_i x_i$$

$$f(m_2)^3 = (\sum y_i)^3 \\ = \sum y_i^3 + \sum y_i^2 y_j + \sum y_i y_j y_k + \sum y_i y_j^2 + \dots$$

We do have a map

$$\chi: \text{NCSym} \rightarrow \text{Sym} \\ (y_i \mapsto x_i)$$

$$\chi(\sum_{i,j} x_i x_j x_i) = \sum_{i,j} x_i^2 x_j$$

which forgets that the variables are non-comm.

We also have a map

$$\psi: \text{NSym} \rightarrow \text{Sym}^* \\ h_i \mapsto h_i$$

$$\chi(h_2 h_3 h_2) = h_3 h_2$$

which forgets that the  $h_i$  are non-comm.

These are clearly surjective, so

$\text{Sym}$  is a quotient of  $\text{NSym}, \text{NCSym}$

$NSym$  and  $NCSym$  are also related.

The map

$$\chi: NCSym \rightarrow Sym$$

has a "pullback"

$$\begin{array}{c} \tilde{\chi}: Sym \rightarrow NCSym \\ m_\lambda \mapsto \sum_{\mu \in \mathcal{D}(\lambda)} m_\mu \end{array} \quad \tilde{\chi} \left( \sum_{i < j} x_i^2 x_j \right) = \sum_{i < j} x_i^2 x_j + \sum_{i < j} x_i x_j x_i + \sum_{i < j} x_i x_j^2$$

such that

$$\chi \circ \tilde{\chi} = \text{identity}$$

This gives

$$\begin{array}{c} I: NSym \hookrightarrow NCSym \\ h_n \mapsto \tilde{\chi}(h_n) \end{array}$$

$$\begin{array}{ccc} NSym & \xrightarrow{I} & NCSym \\ \downarrow \psi & \nearrow \tilde{\chi} & \\ & Sym & \end{array}$$

Then the following diagram commutes:

$$\begin{array}{ccc} NSym & \xrightarrow{I} & NCSym \\ \downarrow \psi & \nearrow \tilde{\chi} & \downarrow \chi \\ Sym & \xlongequal{\quad} & Sym^* \end{array}$$

## (B) $Sym, NSym, QSym, \mathcal{G}Sym$

Aside from  $NSym \downarrow Sym$ , I have a

natural inclusion

$$Sym \hookrightarrow QSym$$

Let's bring  $\mathcal{G}Sym$  into the picture.

Given a permutation  $v \in S_n$ , let the descent set of  $v$  be

$$Des(v) = \{i \in [n-1] : v_i > v_{i+1}\}$$

For example

$$Des(\underline{7} \underline{2} \underline{3} \underline{6} \underline{1} \underline{5} \underline{4}) = \{1, 4, 6\}$$

which gives the composition 1321 of 7

Prop The map

$$D: \mathcal{G}Sym \rightarrow QSym$$

$$F_v \mapsto F_{c(Des(v))}$$

is a morphism of Hopf algebras (Malvenuto)

This is surjective since the  $F_s$  are a basis for  $QSym$ .

Another nice fact:

Thm  $\mathcal{G}\text{Sym}$  is self-dual as a graded Hopf algebra

If  $\{F_U^*\}$  is the basis of  $\mathcal{G}\text{Sym}^*$  dual to  $\{F_U\}$  then an isomorphism is

$$\begin{aligned} \mathcal{G}\text{Sym}^* &\longrightarrow \mathcal{G}\text{Sym} \\ F_U^* &\longmapsto F_{U^{-1}} \end{aligned}$$

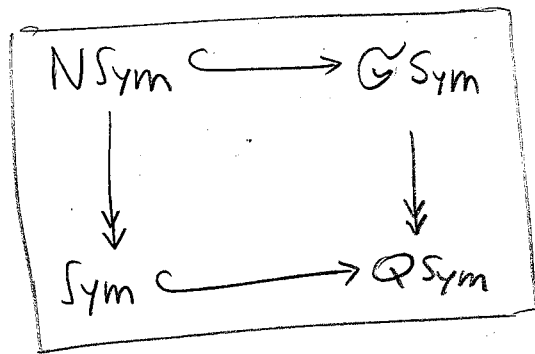
Then

$$\mathcal{G}\text{Sym} \longrightarrow \mathcal{Q}\text{Sym}$$

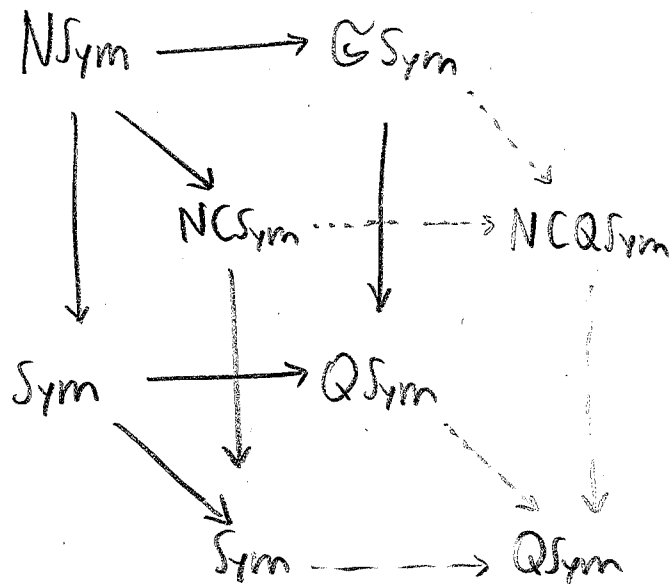
gives a dual map

$$\begin{array}{ccc} \text{NSym} & \longrightarrow & \mathcal{G}\text{Sym} \\ \parallel & & \parallel \mathcal{G} \\ \mathcal{Q}\text{Sym}^* & & \mathcal{G}\text{Sym}^* \end{array} \quad \left( F_I \longmapsto \sum_{\substack{\text{Der}(I) \\ = I}} F_U \right)$$

and we have a commutative diagram



In fact, these two diagrams fit into a larger diagram:



There is so much more to say about this!  
(Much of it open)