Now we discuss some variants of Sym

Lect 27
May 1, 12

Non-commutative Versions

① NCSym

Let $x_1, x_2, \ldots$ be non-commuting indeterminates.

Consider the formal series such as

$$3x_1x_2x_1 + x_1x_1x_3x_3 - 1 + \ldots$$

which are linear combinations of words in $* = (x_1, \ldots)$

Say $f(x)$ is symmetric if permuting the variables leaves it fixed.

Ex: $\sum_{i \neq j} x_i x_j x_i$

Let

$$\boxed{NCSym = \{\text{symm fns in non-comm. vars}\}}$$

(See Rosas-Sagan, Bergeron-Reutenauer-Rosas-Zabrocki)

A <u>set partition</u> π of $[n]$ is $A_1|\cdots|A_k$ where $[n] = A_1 \uplus \cdots \uplus A_k$ and the order of the A_i's is ignored.

Here the <u>monomial symmetric functions</u> are

$$\boxed{m_{14|267|35} = \sum_{i \neq j \neq k} x_i x_j x_k x_i x_k x_j x_j}$$

14 267 35

(def. by example)

Fact.
$\{m_\pi : \pi \text{ set partition}\}$ is a basis for NCSym

Corollary:
$\dim(NCSym_n) = \#$ of set partitions of $[n]$
= "Bell number" B_n

Compare with
$\dim(Sym_n) = \#$ of partitions of $n = P_n$

This is a Hopf algebra with
product: multiplication
coproduct: "$\Delta(f(x)) = f(x,y)$"

② NSym

Consider non-commuting indets. $e_1, e_2, \ldots$ with $\deg e_i = i$.

Let $\boxed{NSym = \mathbb{K}\langle e_1, e_2, \ldots \rangle}$ ← free non-comm algebra gen. by them

A <u>composition</u> is $c = (c_1, c_2, \ldots)$ $c_i \in \mathbb{N}$. Let $e_c = e_{c_1} e_{c_2} \cdots$

<u>Cor.</u> $\dim(NSym_d) = 2^{d-1}$

<u>Cor</u> $\{e_c : c \text{ comp}\}$ is a basis

<u>Pf</u> There are 2^{d-1} comps of d, via the bijection

$$c = (c_1, \ldots, c_\ell) \longrightarrow \{c_1, c_1+c_2, \ldots, c_1+\cdots+c_{\ell-1}\}$$

We also have a coproduct as before
"$\Delta(f(x)) = f(x,y)$"
which satisfies
$$\Delta(e_n) = \sum_{k=0}^{n} e_k \otimes e_{n-k}$$

We can define $h_1, h_2, \ldots$ recursively by
$$\sum_{k=0}^{n} (-1)^k e_k h_{n-k} = \begin{cases} 1 & n=0 \\ 0 & n \geq 1 \end{cases}$$
and $h_c = h_{c_1} h_{c_2} \cdots$.

Then we have

$\boxed{S(e_c) = (-1)^n h_c}$ for a comp. c of n.

Also, $NSym = \mathbb{K}\langle h_1, h_2, \ldots \rangle$, $\Delta(h_n) = \sum_{k=0}^{n} h_k \otimes h_{n-k}$

Quasisymmetric Functions

Let $x_1, x_2, \ldots$ be indeterminates which commute

A function $f(x) = \sum c_\alpha x^\alpha$ is <u>quasisymmetric</u> if
$$x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k} \quad \text{and} \quad x_{j_1}^{\alpha_1} x_{j_2}^{\alpha_2} \cdots x_{j_k}^{\alpha_k}$$
have the same coeff. whenever $i_1 < \cdots < i_k$
$j_1 < \cdots < j_k$

Let $\boxed{QSym = \{\text{quasisymmetric functions}\}}$

Ex:
- $Sym \subset QSym$
- $\sum_{i<j} x_i^2 x_j$ is in $QSym$, not in Sym
- monomial quasisym fns:
$$\boxed{M_\alpha = \sum_{i_1 < \cdots < i_l} x_{i_1}^{\alpha_1} \cdots x_{i_l}^{\alpha_l}} \qquad \alpha = (\alpha_1, \ldots, \alpha_l) \text{ composition}$$
- we can also define elem + complete homog qsym fns, but they are symmetric, so nothing new.

Easy fact:

$\boxed{\{M_\alpha : \alpha \text{ composition}\} \text{ is a basis for } QSym}$ $\qquad$ $\boxed{\dim QSym_d = 2^{d-1}}$

Again we have a coproduct

"$\Delta(f(x)) = f(x,y)$"

and

$$\boxed{\Delta(M_\alpha) = \sum_{\underbrace{\alpha=\beta\gamma}_{\text{concatenation}}} M_\beta \otimes M_\gamma}$$

The antipode is

$$\boxed{S(M_\beta) = (-1)^k \sum_{\alpha \leq \beta^{rev}} M_\alpha}$$

coarsening reverse

Ex: $S(M_{421}) = -(M_{124} + M_{16} + M_{34} + M_7)$

<u>Fact</u> (Hazewinkel, 2001)
As an algebra, QSym is also free

Another useful basis:

The <u>fundamental</u> <u>quasisymmetric</u> <u>function</u> is

$$\boxed{F_\alpha = \sum_{\beta \geq \alpha} M_\beta} \qquad \alpha \models n$$

Clearly

$\boxed{\{F_\alpha : \alpha \text{ composition}\} \text{ is a basis for QSym}}$

Duality of NSym and QSym

We have $\dim NSym_n = \dim QSym_n = 2^{n-1}$.

There is more to this!

We have bases $\{M_\alpha : \alpha \models n\}$ for QSym and $\{h_\alpha : \alpha \models n\}$ for NSym, which give us a pairing

$$QSym_n \times NSym_n \longrightarrow \mathbb{K}$$

$$\langle M_\alpha, h_\beta \rangle = \begin{cases} 1 & \alpha=\beta \\ 0 & \alpha \neq \beta \end{cases}$$

This makes QSym and NSym graded duals as vector spaces. In fact

> <u>Theorem</u> QSym and NSym are graded dual Hopf algebras

<u>Pf</u> Verify that

$$\langle \Delta(h_\alpha), M_\beta \otimes M_\gamma \rangle = \langle h_\alpha, M_\beta M_\gamma \rangle. \quad \blacksquare$$

Also I have natural maps

$Sym \hookrightarrow QSym$ (inclusion)

$NSym \twoheadrightarrow Sym$ (variables commute)

> <u>Prop</u> These maps are dual to each other