Lect 26
Apr 26, 12

Antipode of Sym:

Recall that in HW4 you computed:

$$S(e_n) = \sum_{l\geq 1}\left[(-1)^l \sum_{a_1+\dots+a_l=n} e_{a_1}\cdots e_{a_l}\right]$$

which was a bit unsatisfactory. Now:

$$S(e_3) = -e_3 + e_2e_1 + e_1e_2 - e_1e_1e_1$$
$$= -e_3 + 2e_{21} - e_{111}$$
$$= -m_{111} + 2(m_{21} + 3m_{111}) - (m_3 + 3m_{21} + 6m_{111})$$
$$= -(m_{111} + m_{21} + m_3)$$

Hmm...

Let

$$\boxed{\begin{array}{ll} h_n = \sum_{\lambda \vdash n} m_\lambda & n\in\mathbb{N} \\ h_\lambda = h_{\lambda_1}\cdots h_{\lambda_l} & \lambda \text{ partition}\end{array}}$$

← all deg n monomials

be the homogeneous symmetric functions.

Lemma.

$$\boxed{S(e_n) = (-1)^n h_n}$$

(A much nicer formula!)

Pf Need $S * I = u\epsilon = I * S$. Enough: check for gens $\{e_n\}$.
We need $\sum_k (-1)^k h_k e_{n-k} = \begin{cases} 1 & n=0 \\ 0 & n\geq 1\end{cases}$. (Exercise) ∎

Corollary If $\lambda \vdash n$

$$\boxed{S(e_\lambda) = (-1)^n h_\lambda}$$

→ **Corollary:**

$$S(h_\lambda) = (-1)^n e_\lambda$$

Pf $S^2 = I$ here. ∎

Corollary

$\{h_\lambda : \lambda \vdash n\}$ is a basis for Sym_n

Pf

Spanning: Let $f \in Sym_n$

$$\Rightarrow S(f) = \sum_{\lambda\vdash n} c_\lambda e_\lambda \quad (e_\lambda \text{ basis})$$
$$\Rightarrow f = (-1)^n \sum_{\lambda\vdash n} c_\lambda h_\lambda$$

Lin indep: If $\sum_{\lambda\vdash n} c_\lambda h_\lambda = 0$

then $\sum_{\lambda\vdash n} c_\lambda e_\lambda = 0$ applying S ∎

Corollary

$$\boxed{Sym = \mathbb{K}[h_1, h_2, \dots]}$$

Prop

$$\Delta(h_n) = \sum_{k=0}^{n} h_k \otimes h_{n-k}$$

Pf

$$\Delta(e_n) = \sum_{k=0}^{n} e_k \otimes e_{n-k}$$

$$(-1)^n \Delta(h_n) = \Delta(S(e_n))$$

S flips order of Δ

$$= \sum_{k=0}^{n} S(e_{n-k}) \otimes S(e_k)$$

$$= \sum_{k=0}^{n} h_{n-k} \otimes h_k . \quad \blacksquare$$

Corollary
The map $e_\lambda \mapsto h_\lambda$ is an automorphism of Sym.

In fact we can say more.
There is a natural inner product on Sym_n

$$\langle e_\lambda, h_\mu \rangle = \begin{cases} 1 & \lambda = \mu \\ 0 & \lambda \neq \mu \end{cases}$$

which allows us to identify Sym^* with Sym, regarding h_μ as a linear functional on Sym.

Graded Dual

If $H = \bigoplus_{n \geq 0} H_n$ is a graded Hopf algebra and $\dim(H_n)$ is finite for all n, let

$$H^{gr} = \bigoplus_{n \geq 0} H_n^*$$

be the graded dual of H. It is a Hopf algebra as defined early on in the class.

The inner product on Sym then defines a dual Hopf algebra structure on Sym. But in fact, this precisely sends $e_\lambda \mapsto h_\lambda$.

Thm Sym is a self-dual Hopf algebra.

Pf Unravelling defs, this is $\langle \Delta f, g \otimes h \rangle = \langle f, gh \rangle$.
Enough to show for bases: $f = m_\lambda$, $g = h_\mu$, $h = h_\nu$

$\langle \Delta m_\lambda, h_\mu \otimes h_\nu \rangle =$ $\qquad$ $\langle m_\lambda, h_\mu h_\nu \rangle =$

$=$ coeff of $m_\mu \otimes m_\nu$ in $\Delta(m_\lambda)$ $\qquad$ $= \langle m_\lambda, h_{\mu \cup \nu} \rangle =$

$$\begin{cases} 1 & \lambda = \mu \cup \nu \\ 0 & \text{otherwise} \end{cases} \quad \blacksquare$$