Symmetric Functions

Lect 25
Apr 24, 12

We defined the "<u>Hopf alg. of sym fns</u>" to be:

- generated by $L_0, L_1, L_2, \ldots$
- product: free commutative on L_i's

$$\mathbb{K}[L_0, L_1, L_2, \ldots]$$

- coproduct:

$$\Delta(L_n) = \sum_{k=0}^{n} L_k \otimes L_{n-k}$$

We did these in the context of incidence Hopf algebras; so what is the name about?

Let $\mathbf{x} = (x_1, x_2, \ldots)$ be indeterminates.

For $\alpha = (\alpha_1, \alpha_2, \ldots)$ ($\alpha_i \in \mathbb{N}$, finitely many non-zero)

$$\mathbf{x}^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \ldots$$

> A <u>symmetric function</u> over $\mathbb{K}$ (or a ring R) is
> $$f(\mathbf{x}) = \sum_\alpha c_\alpha \mathbf{x}^\alpha$$
> such that
> $$f(x_{\sigma(1)}, x_{\sigma(2)}, \ldots) = f(x_1, x_2, \ldots)$$
> for any permutation σ of $\mathbb{Z}^+$.

<u>Exs</u>.

- $x_1^2 + x_2^2 + \ldots$
- $(x_1 + x_2 + \ldots)^2$
- $\sum_{i \neq j} x_i^2 x_j$
- $\sum_{i < j} x_i^2 x_j$ ← no!

Let

$$\boxed{Sym = \{\text{symmetric functions}\}}$$

<u>Note</u>:

- Sym is a $\boxed{\mathbb{K}\text{-algebra}}$
  - vector space ✓
  - ring under $+, \cdot$ ✓
- It is graded by degree

Meta-fact: Sym is very important in several areas of mathematics, e.g.:

- representation theory of S_n
- representation theory of GL_n
- cohomology of "flag manifold"

⋮

There are several nice bases of Sym.
Let's study two of them.

A <u>partition</u> is $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ where $\lambda_1 \geq \dots \geq \lambda_\ell \geq$

It is a partition of $n = \sum \lambda_i$ – write $\lambda \vdash n$.

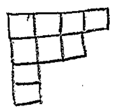
For any partition λ, define the <u>monomial symm fn</u>

$$\boxed{m_\lambda = \sum_{\substack{\text{distinct}\\ \text{permutations}\\ \alpha \text{ of } \lambda}} x^\alpha}$$

Ex: $m_2 = \sum x_i^2$

$m_{311} = \sum_{i \neq j \neq k} x_i^3 x_j x_k$

Draw 4311:



Clearly

$$\boxed{\{m_\lambda : \lambda \vdash n\} \text{ is a basis for } \mathrm{Sym}_n}$$

The <u>elementary symmetric functions</u> are

$$\boxed{\begin{array}{ll} e_n = \sum_{i_1 < \dots < i_n} x_{i_1} \cdots x_{i_n} & (= m_{\underbrace{1111}_{n}}) \\ e_\lambda = e_{\lambda_1} \cdots e_{\lambda_\ell} & \lambda = (\lambda_1, \dots \lambda_\ell) \end{array}}$$

Ex: $e_{21} = \left(\sum_{i \neq j} x_i x_j\right)\left(\sum_k x_k\right) = 3\sum_{i \neq j \neq k} x_i x_j x_k + \sum_{i \neq j} x_i^2 x_j = m_{21} + 3m_{111}$

$$\begin{aligned} e_{111} &= m_3 + 3m_{21} + 6m_{111} \\ e_{21} &= \phantom{m_3 + {}} m_{21} + 3m_{111} \\ e_3 &= \phantom{m_3 + m_{21} + 3{}} m_{111} \end{aligned} \quad \longrightarrow \quad \begin{aligned} m_{111} &= e_3 \\ m_{21} &= -3e_3 + e_{21} \\ m_3 &= 3e_3 - 3e_{21} + e_{111} \end{aligned}$$

<u>Prop.</u> The transition matrix to express $\{e_\lambda\}$ in terms of $\{m_\lambda\}$ is upper triangular, with 1s on the diagonal

(for an appropriate order of rows+cols)

<u>Cor</u> $\boxed{\{e_\lambda : \lambda \vdash n\} \text{ is a basis for } \mathrm{Sym}_n}$

"Fundamental Theorem of Symmetric Functions"

(not the best name...)

Sym is generated by $\{e_1, e_2, e_3, \dots\}$ freely commutatively, so

$$\mathrm{Sym} = \mathbb{K}[e_1, e_2, \dots]$$

<u>Pf:</u> If $\{e_i\}$ satisfied a relation, such as

$$e_4 e_3 = e_5 e_1^2 - 2e_2^3 e_1 + e_7,$$

we would get

$$e_{43} - e_{511} + 2e_{2221} - e_7 = 0 \quad \blacksquare$$

So the algebra structure is as we defined it.

How about the coalgebra?

We define a $\boxed{\text{coproduct}}$ on Sym by

" $\Delta(f(x)) := f(x,y)$ "

More precisely,

For $f(x) = f(x_1, x_2, \ldots)$ symmetric, say

$$f(x,y) = \sum f_{(1)}(x) f_{(2)}(y)$$

Then define

$$\Delta(f) = \sum f_{(1)} \otimes f_{(2)}$$

Ex. $\Delta(m_{21}) = ?$

$$m_{21}(x,y) = \sum_{i \neq j} z_i^2 z_j \qquad z = (x,y)$$

$$= \sum_{i\neq j} x_i^2 x_j + \sum_{i \neq j} y_i^2 y_j + \sum_{i,j} x_i^2 y_j + \sum_{i \neq j} y_i^2 x_j$$

$$= \left(\sum_{i\neq j} x_i^2 x_j\right)(1) + (1)\left(\sum_{i\neq j} y_i^2 y_j\right) + \left(\sum_i x_i^2\right)\left(\sum_j y_j\right) + \left(\sum_i x_i\right)\left(\sum_j y_j^2\right)$$

So

$$\Delta(m_{21}) = m_{21} \otimes 1 + 1 \otimes m_{21} + m_2 \otimes m_1 + m_1 \otimes m_2$$

Similarly,

$$\boxed{\Delta(m_\lambda) = \sum_{\mu \cup \nu = \lambda} m_\mu \otimes m_\nu}$$

Prop $\Delta e_n = \sum_{k=0}^{n} e_k \otimes e_{n-k}$

Pf. $e_n(x,y) = \sum_{i_1 < \cdots < i_n} z_{i_1} \cdots z_{i_n}$ $\qquad z = (x,y)$

$$= \sum_{k=0}^{n} \left(\sum_{a_1 < \cdots < a_k} x_{a_1} \cdots x_{a_k}\right)\left(\sum_{b_1 < \cdots < b_{n-k}} y_{b_1} \cdots y_{b_{n-k}}\right)$$

$$= \sum_{k=0}^{n} e_k(x)\, e_{n-k}(y) \qquad \blacksquare$$

Corollary

Our two definitions of the Hopf algebra of symmetric functions Sym agree

This is only the beginning of a very interesting story. For more, come to Colombia!

ECCO 2012
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Nantel Bergeron will teach a minicourse about this.