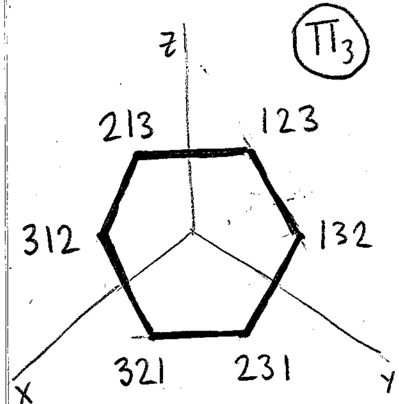
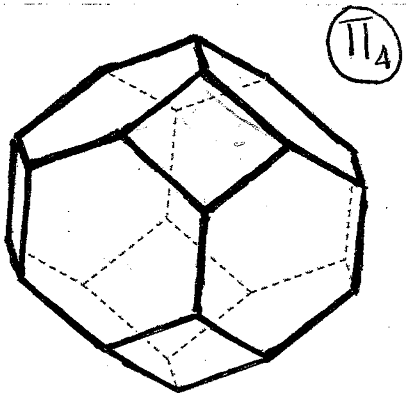
Generalized Permutohedra

The permutohedron $\Pi_n \subset \mathbb{R}^n$ is the polytope

$$\boxed{\Pi_n := \operatorname{conv}\{(\sigma(1), \ldots, \sigma(n)) : \sigma \text{ permutation of } [n]\}}$$

Examples:

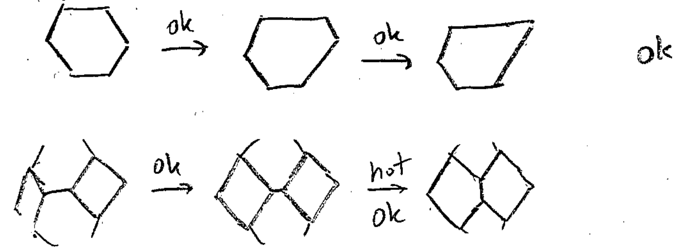




Inequality description:

- $\sum_{i=1}^{n} x_i = 1+2+\cdots+n = \frac{n(n+1)}{2}$
- $\sum_{i\in A} x_i \leq n+(n-1)+\cdots+(n-|A|+1)$

A generalized permutohedron is a polytope obtained by moving the facets (in parallel) without them going across vertices:



Prop/Def. Generalized permutohedra are precisely the polytopes $GP(z)$ of the form

- $\sum_{i\in E} x_i = z(E)$ $\quad (E = \{1, \ldots, n\})$
- $\sum_{i\in A} x_i \leq z(A)$

for a submodular function $z: 2^E \to \mathbb{R} \cup \{\infty\}$

i.e, one such that

$$z(A) + z(B) \geq z(A\cap B) + z(A\cup B)$$

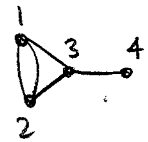
for all $A, B \subseteq E$

Ex 1 (Graphs)

Given a graph $G=(V,E)$, let

$$z_G : 2^V \to \mathbb{R}$$

$z_G(W)$ = # of edges incident to a vertex in W

$G =$ 

$z(\emptyset)=0$
$z(1)=z(2)=z(3)=3$, $z(4)=1$
$z(12)=4$, $z(13)=5, \dots$
$z(123)=5, \dots$
$z(1234)=5$

Exercise: z_G is submodular

$\boxed{GP(z_G) \text{ is the } \underline{\text{graphical zonotope}} \text{ of } G}$

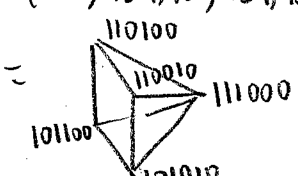
Ex 2 (Matroids)

The matroid polytope of a matroid M is

$$P_M = \mathrm{conv}(e_B : B \text{ basis of } M) \subset \mathbb{R}^E$$

where if $B=\{b_1,\dots,b_r\}$, $e_B = 00\underset{b_1}{1}0\underset{b_2}{1}0\dots 0100\underset{b_r}{1}0$

Ex $\mathcal{B}=\{123, 124, 125, 134, 135\}$

$P_M =$ 

$\subset \mathbb{R}^6$

This was first considered by Edmonds to do optimization on matroids.

$\boxed{\text{Prop } P_M \text{ is a generalized permutohedron}}$

Ex 3 (Posets)

Given a poset P, let $z_P : 2^P \to \mathbb{R}\cup\{\infty\}$ be

$$z_P(A) = \begin{cases} 0 & \text{if } A \text{ is a downset of } P \\ \infty & \text{otherwise} \end{cases}$$

Exercise: z_P is submodular

$\boxed{GP(z_P) \text{ is the } \underline{\text{poset polyhedron}} \text{ of } P}$

The Hopf algebra GP:

Let $P = GP(z)$ be a gen. perm. in $\mathbb{R}^E$.
$P' = GP(z')$ $\mathbb{R}^{E'}$

Then $\boxed{P\times P'}$ is a gen perm in $\mathbb{R}^{E\sqcup E'}$, with

$$\overline{z}(A\sqcup A') = z(A) + z'(A')$$

so GP is closed under $\times$.

Now I want restriction, contraction.

Let $P = GP(z)$ in $\mathbb{R}^E$

Let $A \subset E$.

Consider the face

$$\boxed{P_A = \{p \in P : p \cdot e_A \text{ is maximal}\}}$$

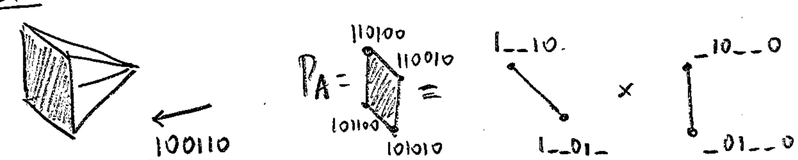
Facts:

- $\boxed{P_A = Q \times R}$ for $Q \subset \mathbb{R}^A$ $R \subset \mathbb{R}^{E-A}$

  Let $\boxed{\begin{array}{l} P|A := Q \\ P/A := R \end{array}}$ be the restriction, contraction

- $P|A$, P/A are gen. perm.

Ex:



$A = \{1,4,5\}$

$= P|A \times P/A$

Theorem (Aguiar-Ardila)

The product $P \cdot P' := P \times P'$

and coproduct $\Delta(P) = \sum_{A \subseteq E} (P|A) \otimes (P/A)$

give a Hopf algebra of generalized permutohedra

We can specialize to graphs, matroids, posets

Gen. perm. P	P|A	P/A	
$GP(Z_G)$	$GP(Z_{G\|A})$	$GP(Z_{G\|V-A})$	
P_M	$P_{M\|A}$	$P_{M/A}$	
$GP(Z_P)$	$GP(Z_{P\|A})$	$GP(Z_{P\|A})$	A order ideal
	0	0	otherwise

Corollary

The Hopf algebras of graphs, matroids, and posets are subalgebras of the Hopf alg. of gen. perm.

Theorem (Aguiar-Ardila)

The antipode for generalized permutohedra is

$$S(P) = \sum_{\substack{Q \text{ face} \\ \text{of } P}} (-1)^{\dim P - \dim Q} Q$$

Corollaries:

- Humpert-Martin formula for $S(G)$
- New formula for $S(M)$
- Schmitt formula for $S(P)$.