(Face lattices of polytopes)

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Let P be a polytope

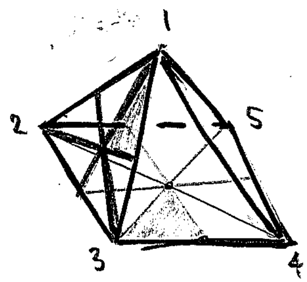
Let L_P be the poset of faces (by inclusion)

Note L_P is a lattice with $F \wedge G = F \cap G$.

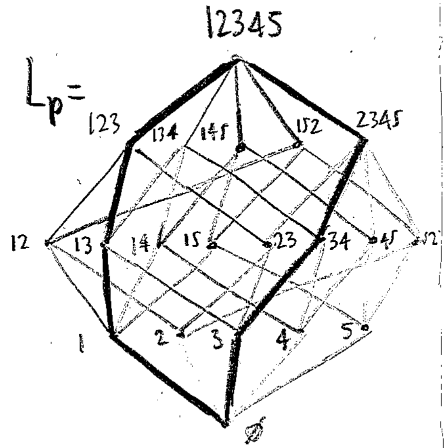
$$\boxed{\textbf{Prop}\quad \mu(L_P) = (-1)^{\dim P}}$$

Pf by example.

$P =$



$L_P =$



Perform the barycentric subdivision ∂P_{bar} of the boundary ∂P of P. (Shown partially) Put a vertex at the barycenter of each face, and connecting barycenters corresponding to chains of faces

Fact: The barycentric subdivision is (a realization of) $\Delta(\bar{P})$

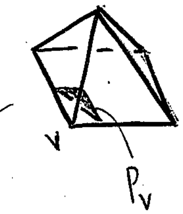
So $\mu(P) = \bar{\chi}(\Delta(\bar{P})) = \bar{\chi}(\partial P_{bar}) = \bar{\chi}(\partial P) = (-1)^{\dim P}$. ∎

$\bar{\chi}$ is topological ↑   ↑ Euler

In fact,

$\{$face lattices of polytopes$\}$

is a hereditary family.



Intervals:

- $[\hat{0}, F]$ is the face lattice of F
- $[v, \hat{1}]$ is the face lattice of the vertex figure P_v if v is a vertex, then induct

Products

- $L_P \times L_Q$ is the face poset of $P \times Q = \{(p,q) : p \in P, q \in Q\}$

So
- they form a Hopf algebra
- $\boxed{\mu_{L_P}(F, G) = (-1)^{\dim G - \dim F}}$ for $F \subseteq G$ faces of P.

This is very useful when counting/measuring things related to polytopes — for example in Ehrhart theory.

An application: Dehn-Sommerville relations

The **f-vector** $(f_{-1}, f_0, f_1, \dots, f_{d-1})$ of a polytope is $f_i = \#$ of i-dim faces of P

The **h-vector** $(h_0, h_1, \dots, h_d)$ is given by

$$\sum_{i=0}^{d} h_i x^{d-i} = \sum_{i=0}^{d} f_{i-1} (x-1)^{d-i}$$

Ex  $f = (1,6,12,8) \to h = (1,3,3,1)$

Stanley's trick:

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 3 & & & \\
 & & 1 & & 4 & & 3 & & \\
 & 1 & & 5 & & 7 & & 1 & \leftarrow \text{h-vector} \\
1 & & 6 & & 12 & & 8 & & \leftarrow \text{f-vector}
\end{array}$$

P is **simplicial** if every facet is a simplex (and hence every face is a simplex)

Thm (Dehn-Sommerville)
If P is simplicial then $\boxed{h_i = h_{d-i}}$ for all i.

Also, these are the only equations satisfied by any f-vector of any simplicial polytope.

In "f-language", $h_i = h_{d-i}$ translates to

$$f_k \overset{?}{=} \sum_{\ell=k}^{d} (-1)^{d-\ell+1} \binom{\ell+1}{k+1} f_\ell$$

We have

$$\begin{aligned}
(-1)^{d-k} f_k &= \sum_{\dim F = k} \mu(F,P) \\
&= \sum_{\dim F = k} \Big(- \sum_{F \le G < P} \mu(F,G) \Big) \\
&= - \sum_{\substack{G < P \\ \dim G \ge k}} \sum_{\substack{F \le G \\ \dim F = k}} (-1)^{\dim G - \dim F} \\
&= - \sum_{\substack{G < P \\ \dim G \ge k}} \binom{\dim G + 1}{k+1} (-1)^{\dim G - k} \\
&= \sum_{\ell \ge k} (-1)^{\ell - k + 1} \binom{\ell+1}{k+1} f_\ell . \quad \blacksquare
\end{aligned}$$