Möbius Inversion

PF HW3

Let $f, g: P \to \mathbb{F}$ be functions.

$$\left(g(x) = \sum_{y \le x} f(y) \text{ for all } x \right) \iff \left(f(x) = \sum_{y \le x} \mu(y,x) g(y) \text{ for all } x \right)$$

(and dually with $y \ge x$, $\mu(x,y)$)

Examples (just a few of *many*)

"Inclusion-Exclusion"

S = set of properties that "people" can have

$A \subseteq S \Rightarrow g(A)$ = # people having properties in A

$f(A)$ = # people having properties in A and no others.

Then

$$g(A) = \sum_{B \supseteq A} f(B) \Rightarrow f(A) = \sum_{B \supseteq A} \mu(A,B) g(B)$$

$$\boxed{f(A) = \sum_{B \supseteq A} (-1)^{|B-A|} g(B)}$$

Typical GRE question: $S = \{\text{blonde, soccer player, female}\}$

This is useful in many, many situations.

An example:

Ex "people" = permutations of $[n]$

$S = \{S_1, \dots, S_n\}$ $\quad S_i$: property that $\pi(i) = i$

For $T \subseteq S$, $g(T) = (n - |T|)!$ keep elts of T fixed, shuffle the rest

$$\Rightarrow \left(\begin{array}{c}\text{number of} \\ \text{"derangements"} \\ \text{of } [n]\end{array}\right) = f(\emptyset) = \sum_{B \supseteq \emptyset} (-1)^{|B|} g(B)$$

$$= \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)!$$

$$= n! \sum_{k=0}^{n} \frac{(-1)^k}{k!} \approx \frac{n!}{e}$$

Lect 21 Apr 10, 12

"Euler's totient function"

$\varphi(n)$ = # of integers $1 \le k \le n$ with $(n,k) = 1$.

Let

$f(m)$ = # of integers $1 \le k \le n$ with $(n,k) = m$

$g(m)$ = # of integers $1 \le k \le n$ with (n,k) a multiple of m

$\quad$ = # of integers $1 \le k \le n$ with $m \mid k$ $= n/m$

We have

$$g(m) = \sum_{\substack{l \ge m \\ \text{in } D_n}} f(l) \Rightarrow f(m) = \sum_{l \ge m} \mu(m,l) g(l)$$

$$= \sum_{l \ge m} \mu(l/m) \, n/l$$

So

$$f(1) = \boxed{\varphi(n) = \sum_{l \mid n} \mu(l) \frac{n}{l} = n\left(1 - \frac{1}{p_1}\right) \cdots \left(1 - \frac{1}{p_t}\right)} \qquad n = \prod p_i^{\alpha_i}$$

Topological Interpretation of Möbius numbers

Theorem (Philip Hall)

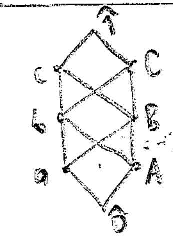
Let P be a poset with $\hat{0}$ and $\hat{1}$.

Let

C_i = # of chains of length i from $\hat{0}$ to $\hat{1}$ in P

Then $\boxed{\mu(P) = C_0 - C_1 + C_2 - C_3 + \cdots}$

Ex



$C_0 = 0$

$C_1 = 1$

$C_2 = 6$

$C_3 = \binom{3}{2} 2^2 = 12$

$C_4 = 2^3 = 8$

$\mu(P) = 0 - 1 + 6 - 12 + 8$

$\mu(P) = 1$

Pf Compute in the incidence algebra

$$\begin{aligned}
\mu(\hat{0},\hat{1}) &= \zeta^{-1}(\hat{0},\hat{1}) \\
&= (1-(1-\zeta))^{-1}(\hat{0},\hat{1}) \\
&= \sum_{k\geq 0} (1-\zeta)^k(\hat{0},\hat{1}) \\
&= \sum_{k\geq 0} (-1)^k(\zeta-1)^k(\hat{0},\hat{1}) \\
&= \sum_{k\geq 0} (-1)^k C_k
\end{aligned}$$

HW 2

∎

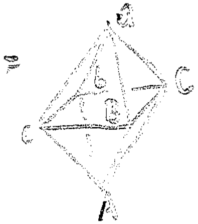
Let $\bar{P} = P - \{\hat{0},\hat{1}\}$

The order complex $\Delta(\bar{P})$ is a simplicial complex

- vertices: elements of $\bar{P}$
- faces/simplices: chains of $\bar{P}$

In the example,

$\Delta(\bar{P}) =$



solid triangles
hollow interior

The reduced Euler characteristic of a simplicial complex Δ is

$$\tilde{\chi}(\Delta) = \sum_i (-1)^i f_i$$

where f_i = # of i-dim faces (and $f_{-1} = 1$, the "empty face")

So:

Theorem
$$\boxed{\mu(P) = \tilde{\chi}(\Delta(\bar{P}))}$$

which allows us to use topology to compute Euler characteristics.