Corollary

The antipode of the Faà di Bruno Hopf alg is

$$S(x_n) = \sum_{k\geq 1} (-1)^k B_{n+k,k}(0, x_1, x_2, \ldots)$$

Pf

Compute (twice) the inverse of $F(f) = \sum_{n\geq 1} f(x_{n-1}) \frac{t^n}{n!}$.

- It is $F(f^{-1}) = F(f \circ S) = \sum_{n\geq 1} f(S(x_{n-1})) \frac{t^n}{n!}$.
  (f^{-1} in $\mathcal{X}(\mathcal{F})$)

- It is $\sum_{n\geq 1} g_{n-1} \frac{x^n}{n!}$ where, by Lagrange inversion,

$$g_n = \sum_{k\geq 1} (-1)^k B_{n+k,k}(0, f(x_1), f(x_2), \ldots)$$

$$= f\left(\sum_{k\geq 1} (-1)^k B_{n+k,k}(0, x_1, x_2, \ldots)\right)$$

Compare coeffs of $\frac{t^{n+1}}{(n+1)!}$. They are equal for all f, so the result holds. ∎

So $\mathcal{F}$ is an algebraic encoding of composition of $\sum_{n\geq 1} a_n \frac{t^n}{n!}$:

(formula for antipode of $\mathcal{F}$) $\longleftrightarrow$ (formula for $f^{-1}(x)$)

(formula for coproduct of $\mathcal{F}$) $\longleftrightarrow$ (formula for $f(g(x))$)

([Haiman-Schmitt] prove the formula for $S(x_n)$ combinatorially, and use it to deduce Lagrange inversion)

From antipodes to Möbius functions

In the incidence algebra of a(ll) poset(s) we defined

$$1([x,y]) = \begin{cases} 1 & x = y \\ 0 & \text{otherwise} \end{cases} \quad \text{(identity under convolution)}$$

$$\zeta([x,y]) = 1 \quad \text{for all } [x,y]$$

$\mu = \zeta^{-1}$ in the convolution algebra

Prop $\mu = \zeta \circ S$ (ζ is an alg map)

Pf: $\zeta * (\zeta \circ S) = \zeta(I) * \zeta(S) =$

$= \zeta(I * S) = \zeta u \epsilon$

On a trivial poset, $\zeta u \epsilon(P) = \zeta u(1) = 1$

nontrivial $\zeta u \epsilon(P) = \zeta u(0) = 0$

So $\zeta * (\zeta \circ S) = 1$ ∎

So a formula for S gives a formula for μ

For the Faà di Bruno algebra we can say even more

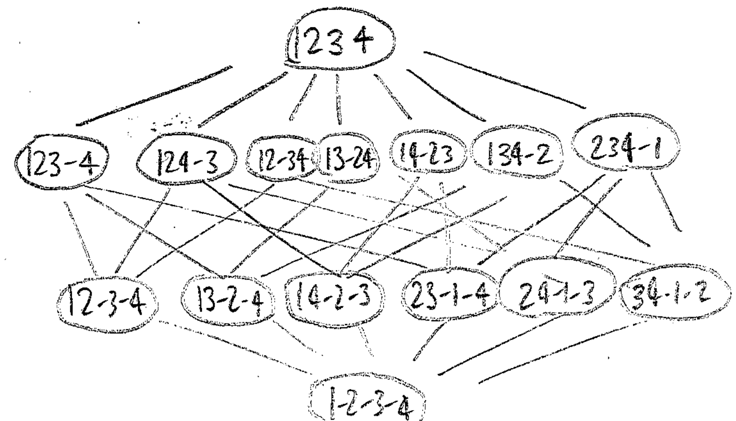
Lect 20
Apr 3

Let Π_n = lattice of flats of K_n

= poset of "set partitions" of $[n]$ ordered by refinement.

= "partition lattice"

$\Pi_4 =$



Prop $\mu(\Pi_n) = (-1)^{n-1}(n-1)!$

Pf We have $\mu * I = 1$ in $\mathbb{X}(\mathcal{F})$ (these are characters of $\mathcal{F}$) so in power series

$F(I) \circ F(\mu) = F(1)$

Now, $F(I) = \sum_{n\geq 1} I(X_{n-1})\frac{t^n}{n!} = e^t - 1$, $F(1) = \frac{t^1}{1!} = t$

So $F(\mu) = \log(1+t) = \int \frac{1}{1+t}\,dt$

$\sum_{n\geq 1} \mu(X_{n-1})\frac{t^n}{n!} = \sum_{n\geq 1}(-1)^{n-1}\frac{t^n}{n}$

Möbius Inversion

The Möbius function of a poset P is

$\mu : \mathrm{Int}(P) \longrightarrow \mathbb{K}$ defined by $\mu * \zeta = 1$

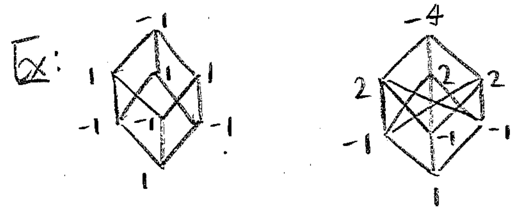
which we can rewrite as $\sum_{z : x \leq z \leq y} \mu(x,z) = \begin{cases} 1 & x = y \\ 0 & x < y \end{cases}$

So we can also define it recursively:

$$\mu(x,x) = 1$$
$$\mu(x,y) = -\sum_{z : x \leq z < y} \mu(x,z) \qquad x<y$$

Often we are only interested in $\mu(x) = \mu(\hat{0}, x)$.

Ex:



$q^3 - 3q^2 + 3q - 1$ $\qquad$ $q^3 - 3q^2 + 6q - 4$

If P is ranked with $\hat{0}$, the characteristic polynomial is

$$\chi_P(q) = \sum_{x \in P} \mu(\hat{0}, x)\, q^{rk(P) - rk(x)}$$

If P has a $\hat{0}$ and a $\hat{1}$, the Möbius number is

$$\mu(P) = \mu(\hat{0}, \hat{1})$$

Ex $P = B_n =$ 

$\mu(S,T) = ?$ $\qquad [S,T] \cong B_{|T-S|}$

$$\boxed{\mu(S,T) = \mu(B_k) \qquad k = |T-S|}$$

In the Hopf algebra of $(\{\text{posets}\}/\text{isom})$, the subalg. gen by Boolean posets was the binomial Hopf algebra. We had

$$S(B_k) = (-1)^k B_k$$

Therefore

$$\mu(B_k) = (\zeta \circ S)(B_k) = \zeta((-1)^k B_k) = (-1)^k$$

$$\boxed{\mu(B_k) = (-1)^k}$$

(Can also prove by induction, this is cleaner.)

Ex $P = L_n =$ 

$$\boxed{\mu(L_n) = \begin{cases} 1 & n=0 \\ -1 & n=1 \\ 0 & n \geq 1 \end{cases}}$$

The subalg gen by L_n was the Hopf alg of symm fns. We had (HW)

$$S(L_n) = \sum_{\substack{p_1 > \dots > p_k \geq 1 \\ \alpha_1, \dots, \alpha_k \geq 1 \\ \Sigma \alpha_i p_i = n}} (-1)^{\Sigma \alpha_i} \binom{\Sigma \alpha_i}{\alpha_1, \dots, \alpha_k} L_{p_1}^{\alpha_1} \cdots L_{p_k}^{\alpha_k}$$

so

$$\mu(L_n) = \sum_{\substack{p_1 > \dots > p_k \geq 1 \\ \alpha_i \geq 1 \quad \Sigma \alpha_i p_i = n}} (-1)^{\Sigma \alpha_i} \binom{\Sigma \alpha_i}{\alpha_1 \dots \alpha_k} = \dots = \begin{cases} 1 & n=0 \\ -1 & n=1 \\ 0 & n \geq 1 \end{cases}$$

not so efficient

Ex $P = D_n = \{\text{divisors of } n\}$

$\mu(a,b) = ?$ $\qquad [a,b] \cong D_{b/a}$

$$\boxed{\mu(a,b) = \mu(D_k) \qquad k = b/a}$$

Sup $k = p_1^{\alpha_1} \cdots p_i^{\alpha_i}$.

Then $D_k \cong L_{\alpha_1} \times \dots \times L_{\alpha_i}$ (check)

so $\mu(D_k) = \mu(L_{\alpha_1}) \cdots \mu(L_{\alpha_i})$

$$\boxed{\mu(D_k) = \begin{cases} 0 & \text{if some } \alpha_i \geq 2 \\ 1 & \text{if } k = p_1 \dots p_i \quad (i \text{ even}) \\ -1 & \text{if } k = p_1 \dots p_i \quad (i \text{ odd}) \end{cases}}$$

Here we are using

> **Prop** $\mu(P \times Q) = \mu(P)\mu(Q)$
>
> More generally, $\mu_{P \times Q}((p,q),(p',q')) = \mu_P(p,p')\mu_Q(q,q')$

Pf This follows from $S(P \times Q) = S(Q)S(P)$ ∎

Ex $P = J(Q)$ $\qquad$ distributive lattice

$\mu(I,I') = ?$ $\qquad [I,I'] \cong J(I'-I)$

$$\boxed{\mu(I,I') = \mu(J(R)) \qquad R = I'-I}$$

Fact: (not obvious)

$$\mu(J(R)) = \begin{cases} (-1)^{|R|} & R = \circ\ \circ\ \circ\ \circ \quad (\text{antichain}) \\ 0 & \text{otherwise} \end{cases}$$

Möbius Inversion

PF HW3

Let $f, g: P \to \mathbb{F}$ be functions.

$$\left(g(x) = \sum_{y \le x} f(y) \text{ for all } x \right) \iff \left(f(x) = \sum_{y \le x} \mu(y,x) g(y) \text{ for all } x \right)$$

(and dually with $y \ge x$, $\mu(x,y)$)

Examples. (just a few of *many*)

"Inclusion-Exclusion"

S = set of properties that "people" can have

$A \subseteq S \Rightarrow g(A)$ = # people having properties in A

$f(A)$ = # people having properties in A and no others.

Then

$$g(A) = \sum_{B \supseteq A} f(B) \Rightarrow f(A) = \sum_{B \supseteq A} \mu(A,B)\, g(B)$$

$$\boxed{f(A) = \sum_{B \supseteq A} (-1)^{|B-A|} g(B)}$$

Typical GRE question: S = {blonde, soccer player, female}

This is useful in many, many situations.

An example:

Ex "people" = permutations of $[n]$

$S = \{S_1, \dots, S_n\}$ S_i: property that $\pi(i) = i$

For $T \subseteq S$, $g(T) = (n - |T|)!$ keep elts of T fixed, shuffle the rest

$$\Rightarrow \left(\text{number of "derangements" of } [n]\right) = f(\emptyset) = \sum_{B \supseteq \emptyset} (-1)^{|B|} g(B)$$

$$= \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)!$$

$$= n! \sum_{k=0}^{n} \frac{(-1)^k}{k!} \approx \frac{n!}{e}$$

"Euler's totient function"

$\varphi(n)$ = # of integers $1 \le k \le n$ with $(n,k) = 1$.

Let

$f(m)$ = # of integers $1 \le k \le n$ with $(n,k) = m$

$g(m)$ = # of integers $1 \le k \le n$ with (n,k) a multiple of m

= # of integers $1 \le k \le n$ with $m \mid k$ $= n/m$

We have

$$g(m) = \sum_{\substack{l \ge m \\ \text{in } D_n}} f(l) \Rightarrow f(m) = \sum_{l \ge m} \mu(m,l)\, g(l)$$

$$= \sum_{l \ge m} \mu(l/m)\, n/l$$

So

$$f(1) = \boxed{\varphi(n) = \sum_{l \mid n} \mu(l) \frac{n}{l} = n\left(1 - \frac{1}{p_1}\right) \cdots \left(1 - \frac{1}{p_t}\right)}$$

$n = \prod p_i^{\alpha_i}$