Prop In any incidence Hopf algebra, $S \circ S = I$

(Note: these need not be comm/cocomm.)

Pf HW4

Remark

Suppose P is a family of intervals closed under taking intervals, such that $P, Q \in P \Rightarrow P \times Q \notin P$.

Then we can let

$$P^* = \{P_1 \times \dots \times P_n : P_i \in P\}$$

and this is a hereditary family.

(Also if we have $\sim$ on P, get $\sim$ on P^*)

$\Rightarrow H(P^*) =$ free commutative incidence Hopf algebra of P

Ex 1 (Binomial Hopf algebra)

Let $\mathcal{B} = \{\text{finite Boolean algebras}\}$ $\qquad \boxed{H(\mathcal{B}) \cong \mathbb{K}[x]}$

$\sim$ = isomorphism

Then $\mathcal{B}/\sim = \{B_0, B_1, B_2, B_3, \dots\}$ so $H(\mathcal{B}) = \mathbb{K}\{B_0, B_1, B_2, \dots\}$

Product: $B_i \cdot B_j = B_i \times B_j = B_{i+j}$

Coproduct: $\Delta(B_i) = \sum_{S \subset [i]} [\emptyset, S] \otimes [S, [i]] = \sum_{k=0}^{i} \binom{i}{k} B_k \otimes B_{i-k}$

Antipode: $S(B_1) = (-1)^1 [\emptyset, \{1\}] = (-1)^1 B_1 \Rightarrow S(B_n) = (-1)^n B_n$

Unit, counit. ✓ $\qquad (S(xy) = S(x)S(y))$

Ex 2. (Hopf algebra of symmetric functions)

Let $\mathcal{L} = \{\text{finite linear orders}\}$

$\sim$ = isomorphism

Then $\mathcal{L}/\sim = \{L_0, L_1, L_2, \dots\}$ $\qquad H(\mathcal{L}) = \mathbb{K}[L_0, L_1, L_2, \dots]$

This is not hereditary, but $\mathcal{L}^* = \{\text{finite products of finite linear orders}\}$ is.

Product: free commutative

Coproduct: $\Delta(L_n) = \sum_{k=0}^{n} L_k \otimes L_{n-k} \qquad n \geq 0$

Antipode: HW 4

Unit: $u(1) = L_0$

Counit: $\epsilon(L_n) = \begin{cases} 1 & n=0 \\ 0 & n \geq 1 \end{cases}$ $\qquad$ Grading: $\deg L_n = n$

Will see: $\boxed{H(\mathcal{L}) \cong Sym}$ "Hopf algebra of symmetric functions"

This is a very important example, we will have much more to say about it.

__Ex 3__ (Hopf algebra of noncomm. symmetric functions)

Regard $\mathbb{Z}^n$ as a poset under componentwise order:

$$(a_1,\dots,a_n) \le (b_1,\dots,b_n) \iff a_i \le b_i \text{ for } 1\le i\le n$$

Let

$\mathcal{B} = \{\text{"finite boxes"}\}$

$= \{ [a,b] : a,b \in \mathbb{Z}^n \text{ for some } n\ge 0, a\le b\}$

This is a hereditary family.

(Note: $\underset{\mathbb{Z}^m}{\underset{\uparrow}{[a,b]}} \times \underset{\mathbb{Z}^n}{\underset{\uparrow}{[c,d]}} = \underset{\mathbb{Z}^{m+n}}{\underset{\uparrow}{[(a,c),(b,d)]}}$

Let $\sim$ identify boxes of the "same shape and orientation":

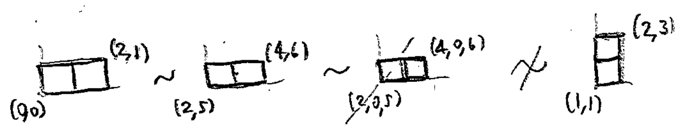
Given a vector $v \in \mathbb{N}^n$, let $\bar{v}$ be $\bar{v}$ with all its 0s omitted.

$$\overline{(0,2,3,0,0,2,0,2,1)} = (2,3,2,2,1)$$

Then let

$$[a,b] \sim [c,d] \iff \overline{b-a} = \overline{d-c}$$

Ex:



This is a "reduced congruence": $\sim$ such that

$$\boxed{\begin{array}{l} P\sim Q \Rightarrow \exists \text{ bijection } f: P\to Q \text{ s.t. } \begin{array}{l}[0,x]\sim[0,f(x)] \\ [x,1]\sim[f(x),1]\end{array} \ \forall x\in P \\ P\sim Q \Rightarrow P\times R \sim Q\times R,\ R\times P\sim R\times Q \\ |Q|=1 \Rightarrow P\times Q\sim P\sim Q\times P \end{array}}$$

Then

$$\mathcal{B}/\sim = \{ B_{(\alpha_1,\dots,\alpha_k)} : \alpha_i \in \mathbb{Z}_{>0}, k\in\mathbb{N}\} \cup \{B_0\}$$

(α is a "composition" of $\alpha_1+\dots+\alpha_k$)

Product: free non-commutative on B_i: $B_{(1,3)}B_{(2,2)} = B_{(1,3,2,2)}$

Coproduct: $\Delta(B_n) = \sum_{i+j=n} B_k \otimes B_{n-k}$, extend multiplicatively

Antipode: HW 4

Unit: $u(1) = B_0$

Counit: $\epsilon(B_\alpha) = \begin{cases} 1 & \alpha = 0 \\ 0 & \text{otherwise}\end{cases}$

Grading: $\deg(B_\alpha) = \alpha_1+\dots+\alpha_k = wt(\alpha)$

Will see: $\boxed{H(\mathcal{B}) \cong NSym}$ "Hopf algebra of noncommutative symmetric functions"

This is another important example we will say much more about.

Ex 4. (Incidence Hopf algebra of graphs)

Want: $H(\mathcal{G}) = \text{span}\{\text{finite simple graphs}\}_{\mathbb{k}}$, $G_1 \cdot G_2 = G_1 \, G_2 = G_1 \uplus G_2$ disjoint union

(from a hereditary family of posets)

Need: graph G on $V \longrightarrow$ poset $P(G)$

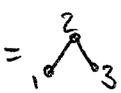
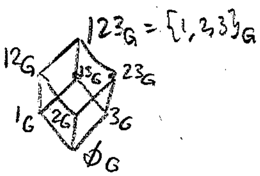
A trivial construction:

graph G on $V \mapsto$ poset $P(G) = 2^V = \{\text{subsets of } V\}$

Problems:

- If G, G' are different graphs on V, how do we tell apart $P(G), P(G')$?

  Sol: label elts of $P(G)$ with a G:

  $G =$  $\longrightarrow P(G) =$ 

- $P(G)$ knows very little about G

  Sol: Use $\sim$

  Let $P(G_1) \sim P(G_2)$ if $G_1 \cong G_2$ as graphs

- $\{P(G) \mid G \text{ graph}\}$ is not closed under taking intervals and direct products.

  Sol: Enlarge it

$$\mathcal{P}(\mathcal{G}) = \left\{ [U_1, W_1]_{G} \times \cdots \times [U_n, W_n]_{G_n} : \begin{array}{l} G_1, \dots, G_n \text{ graphs on } \\ V_1, \dots, V_n \text{ vertex sets} \\ U_i \subseteq W_i \subseteq V_i \end{array} \right\}$$

Let

$$[U_1, W_1]_{G_1} \times \cdots \times [U_n, W_n]_{G_n} \sim [U_1', W_1']_{G_1'} \times \cdots \times [U_n', W_n']_{G_n'}$$

if

$$\biguplus G_i|_{W_i \setminus U_i} \cong \biguplus G_i'|_{W_i' \setminus U_i'}$$

Check: This is a reduced congruence

$\Downarrow$

I get an incidence Hopf algebra of graphs

$$\begin{pmatrix} \sim \text{ classes} \\ \text{of } \mathcal{P}(\mathcal{G}) \end{pmatrix} \xleftrightarrow{\text{bij.}} \begin{pmatrix} \cong \text{ classes} \\ \text{of graphs} \end{pmatrix}$$

$$\prod_i [U_i, W_i]_{G_i} \longmapsto \biguplus_i G_i|_{W_i \setminus V_i}$$

Linear basis: {Isom classes of graphs}

Product: $G \cdot H = G \uplus H$

Coproduct: $\Delta(G) = \sum_{A \subseteq V} G|_A \otimes G|_{V \setminus A}$

$$\Delta([\emptyset, V]_G) = \sum_{A \subseteq V} [\emptyset, A]_G \otimes [A, V]_G$$

Antipode: $S(G) = \sum_{\emptyset \subset V_1 \subsetneq \cdots \subsetneq V_k = V} (-1)^k \prod G|_{V_i \setminus V_{i-1}}$

$$= \sum_{\pi \text{ partition of } V} (-1)^{|\pi|} |\pi|! \prod_{B \in \pi} G|_B \quad \text{(Takeuchi)}$$

(HW3)

$\uparrow$

Optimal Formula: Humpert-Martin "Incidence Hopf Alg of Graphs"