Theorem (Takeuchi '71)

A graded, connected bialgebra H has an antipode. If $\pi = I - u\epsilon : H \to H$ then

$$S = \sum_{n \geq 0} (-1)^n m^{n-1} \pi^{\otimes n} \Delta^{n-1}$$

which turns it into a Hopf algebra.

Convention: $m^0 = \Delta^0 = id$
$m^{-1} = u, \Delta^{-1} = \epsilon$

Note $\Delta^n(H_m) = 0$ for $n > m$
So $\Delta^n(h)$ is a finite sum for any $h \in H$.

Pf Recall that in the convolution product

$$\pi^{*n} = \sum_{(h)} \pi(h_{(1)}) \cdots \pi(h_{(n)})$$

$$\boxed{\pi^{*n} = m^{n-1} \pi^{\otimes n} \Delta^{n-1}}$$

so really

$$S = \sum_{n \geq 0} (-1)^n \pi^{*n}$$

Then

$$S * I = \Big[\sum_{n \geq 0} (-1)^n \pi^{*n}\Big] * (\pi + u\epsilon)$$

$u\epsilon$ is 1 for convolution

$$= \sum_{n \geq 0} (-1)^n \pi^{*(n+1)} + \sum_{n \geq 0} (-1)^n \pi^{*n}$$

$$= \pi^{*0} = u\epsilon$$

Similarly $I * S = u\epsilon$. ∎

In fact, note that this applies for any **conilpotent** bialgebra, where any $h \in H$ satisfies $\Delta^n h = 0$ for some $n \geq 0$.

Remark

- A graded bialgebra H is connected $\iff$ $u\epsilon|_{H_0} = I|_{H_0}$

(This is because we always have
$$\begin{tikzcd} \mathbb{K} \arrow[rr, "\cong"] \arrow[dr, "u"'] & & \mathbb{K} \\ & H \arrow[ur, "\epsilon"'] & \end{tikzcd}$$
- if $H_0 \cong \mathbb{K}$ then $u = \epsilon^{-1}$
- if $\dim H_0 > 1$, $\dim(\mathrm{Im}\, u\epsilon) = 1 < \dim(\mathrm{im}\, I)$)

- On H_n $(n \geq 1)$, $u\epsilon = 0$

So $(I - u\epsilon)(h)$ just drops the H_0 part of h.

→ What we did: in convolution product,

$$S = I^{-1} = (u\epsilon + \pi)^{-1} =$$
$$= (1 + \pi)^{-1}$$
$$= \sum_{n \geq 0} (-1)^n \pi^{*n}$$

← finite sum for any fixed h we plug in

Ex $H = \mathbb{K}[x]$

- $\Delta^{n-1}(x^N) = \sum_{\substack{a_1+\dots+a_n=N \\ a_i \geq 0}} \binom{N}{a_1 \dots a_n} x^{a_1} \otimes \dots \otimes x^{a_n}$

Pf by induction.

- $\pi(x^a) = \begin{cases} x^a & a \geq 1 \\ 0 & a = 0 \end{cases}$

So

$$m^{n-1} \pi^{\otimes n} \Delta^{n-1}(x^N) = \sum_{\substack{a_1+\dots+a_n=N \\ a_i \geq 1}} \binom{N}{a_1 \dots a_n} x^N$$

$$= N! \, x^N \underbrace{\sum_{\substack{a_1+\dots+a_n=N \\ a_i \geq 1}} \frac{1}{a_1! \cdots a_n!}}_{\alpha_n}$$

Note that

$$\alpha_n = \underbrace{[x^N]}_{\text{coeff of } x^N} \left(\frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \right)^n$$

$$= [x^N] \, (e^x - 1)^n$$

Therefore

$$S(x^N) = \sum_{n \geq 0} (-1)^n N! \, x^N [x^N] (e^x - 1)^n$$

$$= N! \, x^N [x^N] \frac{1}{1 + (e^x - 1)} = N! \, x^N \frac{(-1)^N}{N!} = (-x)^N.$$

So it's nice to have Takeuchi's formula, particularly as an existential statement, but this formula isn't always so easy to use / so enlightening.

Other examples in HW3 where Takeuchi's formula for S is unnecessarily complicated.

General Question of Interest:

Find the optimal formula for the antipode of a Hopf algebra

Optimal:
- no cancellation
- no repeated terms.

Let's build up our repertoire of Hopf algebras even further

$\boxed{\text{Incidence Hopf Algebras}}$ (Joni–Rota) (Schmitt)

<u>Hereditary family</u>:

A family of finite <u>intervals</u> (poset with $\hat{0}, \hat{1}$) which is closed under:

- taking intervals
- taking (finite) direct products.

<u>Examples</u>:

- all finite intervals
- all finite graded intervals
- all finite "Boolean algebras" (posets $\cong 2^{[n]}$, some n)
    - $[I, J] \cong 2^{|J-I|}$
    - $2^{[n]} \times 2^{[m]} \cong 2^{[m+n]}$

⋮

<u>Non-example</u>:

- all finite chains

<u>Fact</u>: If $\mathcal{P}$ is a hereditary family, then $\mathbb{K}\mathcal{P}$ is a Hopf algebra with

$$P \cdot Q = P \times Q$$

$$\Delta(P) = \sum_{x \in P} [\hat{0}, x] \otimes [x, \hat{1}]$$

More generally, we may want to identify some intervals. For instance, we could let $\sim$ be isomorphism (or something more general) and put a Hopf structure on $\mathcal{P}/\sim$

We could also identify fewer or more intervals.

<u>Prop.</u>

Let $\mathcal{P}$ be a hereditary family of intervals.

Let $\sim$ be isomorphism (or more generally, a "reduced congruence" – we omit the details)

Then $H(\mathcal{P}) = \mathrm{span}(\mathcal{P}/\sim) = \mathbb{K}\{[P] : P \in \mathcal{P}\}$ is a Hopf algebra with

$$P \cdot Q = P \times Q \qquad u(1) = \text{empty poset}$$

$$\Delta(P) = \sum_{x \in P} [\hat{0}, x] \otimes [x, \hat{1}] \qquad \varepsilon(P) = \begin{cases} 1 & |P| = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$S(P) = \sum_{n \geq 0} \sum_{\hat{0} = x_0 < \dots < x_n = \hat{1}} (-1)^n [x_0, x_1] \times \dots \times [x_{n-1}, x_n]$$

<u>Pf</u>

Bialgebra: We did this already for $\mathcal{P}$ = all finite intervals and it works here in the same way

Antipode: $H(\mathcal{P})$ is nilpotent, apply Takeuchi.

$$m^{n-1} \pi^{\otimes n} \Delta^{n-1}(P)$$

$$\bigotimes [x_{i-1}, x_i] \quad (\hat{0} = x_0 \leq \dots \leq x_n = \hat{1})$$

kills terms with trivial intervals $[x, x]$

turns $\otimes$ into Π

∎