

## Theorem (Takeuchi '71)

A graded, connected bialgebra  $H$  has an antipode. If  $\pi = I - u\epsilon: H \rightarrow H$  then

$$S = \sum_{n \geq 0} (-1)^n m^{n-1} \pi^{\otimes n} \Delta^{n-1}$$

which turns it into a Hopf algebra

Convention:  $m^0 = \Delta^0 = \text{id}$       Note  $\Delta^n(H_m) = 0$  for  $n > m$   
 $m^{-1} = u, \Delta^{-1} = \epsilon$       So  $\Delta^n(h)$  is a finite sum for any  $h \in H$ .

PF Recall that in the convolution product

$$\pi^{\ast n} = \sum_{(h)} \pi(h_{(1)}) \cdots \pi(h_{(n)})$$

$$\pi^{\ast n} = m^{n-1} \pi^{\otimes n} \Delta^{n-1}$$

so really

$$S = \sum_{n \geq 0} (-1)^n \pi^{\ast n}$$

Then

$$S \ast I = \left[ \sum_{n \geq 0} (-1)^n \pi^{\ast n} \right] \ast (\pi + u\epsilon)$$

$$= \sum_{n \geq 0} (-1)^n \pi^{\ast(n+1)} + \sum_{n \geq 0} (-1)^n \pi^{\ast n}$$

$$= \pi^{\ast 0} = u\epsilon$$

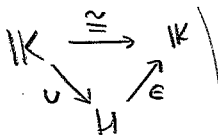
Similarly  $I \ast S = u\epsilon. \blacksquare$

In fact, note that this applies for any co-nilpotent bialgebra, where any  $h \in H$  satisfies  $\Delta^n h = 0$  for some  $n \geq 0$ .

## Remark

- A graded bialgebra  $H$  is connected  $\iff u\epsilon|_{H_0} = I|_{H_0}$

(This is because we always have  
 • if  $H_0 \cong \mathbb{K}$  then  $u = \epsilon^{-1}$   
 • if  $\dim H_0 > 1$ ,  $\dim(\text{Im } u\epsilon) = 1 < \dim(\text{Im } I)$ )



- On  $H_n$  ( $n \geq 1$ ),  $u\epsilon = 0$

So  $(I - u\epsilon)(h)$  just drops the  $H_0$  part of  $h$ .

→ What we did: in convolution product,

$$S = I^{-1} = (u\epsilon + \pi)^{-1} =$$

$$= (1 + \pi)^{-1}$$

$$= \sum_{n \geq 0} (-1)^n \pi^{\ast n}$$

← finite sum for any fixed  $h$  we plug in

$$\underline{E} \quad U = \mathbb{K}[x]$$

$$\bullet \Delta^m(x^N) = \sum_{\substack{a_1 + \dots + a_n = N \\ a_i \geq 0}} \binom{N}{a_1, \dots, a_n} x^{a_1} \otimes \dots \otimes x^{a_n}$$

PF by induction.

$$\bullet \pi(x^a) = \begin{cases} x^a & a \geq 1 \\ 0 & a = 0 \end{cases}$$

So

$$\begin{aligned} m^m \pi^{\otimes n} \Delta^m(x^N) &= \sum_{\substack{a_1 + \dots + a_n = N \\ a_i \geq 1}} \binom{N}{a_1, \dots, a_n} x^N \\ &= N! x^N \sum_{\substack{a_1 + \dots + a_n = N \\ a_i \geq 1}} \frac{1}{a_1! \dots a_n!} \\ &\quad \alpha_n \end{aligned}$$

Note that

$$\begin{aligned} \alpha_n &= [x^N] \left( \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \right)^n \\ &= [x^N] (e^x - 1)^n \end{aligned}$$

Therefore

$$\begin{aligned} S(x^N) &= \sum_{n \geq 0} (-1)^n N! x^N [x^N] (e^x - 1)^n \\ &= N! x^N [x^N] \frac{1}{1 + (e^x - 1)} = N! x^N \frac{(-1)^N}{N!} = (-x)^N \end{aligned}$$

So it's nice to have Takeuchi's formula, particularly as an existential statement, but this formula isn't always so easy to use / so enlightening.

Other examples in HW3, where Takeuchi's formula for  $S$  is unnecessarily complicated.

General Question of Interest:

Find the optimal formula for the antipode of a Hopf algebra

Optimal: - no cancellation  
- no repeated terms.

