Ex 2. $H = \mathbb{K}[x]$

$S(x^n) = (-x)^n$

$$\begin{tikzcd}[column sep=small]
 & \sum_k \binom{n}{k} x^k \otimes x^{n-k} \arrow[r, mapsto] & \sum_k \binom{n}{k} x^k \otimes (-x)^{n-k} \arrow[dr, mapsto] & \\
x^n \arrow[ur, mapsto] \arrow[r, mapsto] \arrow[dr, mapsto] & \begin{cases} 1 & n=0 \\ 0 & n\geq 1 \end{cases} \arrow[rr, mapsto] & & \begin{cases} 1 & n=0 \\ 0 & n\geq 1 \end{cases} \\
 & \sum_k \binom{n}{k} x^k \otimes x^{n-k} \arrow[r, mapsto] & \sum_k \binom{n}{k} (-x)^k \otimes x^{n-k} \arrow[ur, mapsto] &
\end{tikzcd}$$

Ex 3. $H = \mathbb{K}\{$isom. classes of posets with $\hat{0}$ and $\hat{1}\}$

$S(P) = ?$ (Homework)

Ex 4. monoid $G = 2^S$, $A \cdot B := A \cap B$

$H = \mathbb{K}G$

$S(A) = ?$

$$\begin{tikzcd}
 & A \otimes A \arrow[r, mapsto] & A \otimes S(A) \arrow[dr, mapsto, "?"] & \\
A \arrow[ur, mapsto] \arrow[r, mapsto] & 1 \arrow[rr, mapsto] & & 1_G = S
\end{tikzcd}$$

If $S(A) = \sum \lambda_i A_i$ then we need

$$\sum \lambda_i \underbrace{(A_i \cap A)}_{\subseteq A} = S$$

and this is impossible for $A \neq S$.

So this is a bialgebra which cannot be turned into a Hopf algebra, No possible antipode!

A different point of view on S:

Let: C = coalgebra

A = algebra

Consider $\mathrm{Hom}(C, A)$ = $\mathbb{K}$-linear maps from C to A

This is naturally an algebra:

(m) $\mathrm{Hom}(C,A) \otimes \mathrm{Hom}(C,A) \hookrightarrow \mathrm{Hom}(C \otimes C, A \otimes A) \to \mathrm{Hom}(C,A)$

$(f, g) \mapsto f \otimes g(c \otimes c') = f(c) \otimes g(c')$

$\alpha : C \otimes C \to A \otimes A \mapsto \beta : C \to A$

where

$$\begin{tikzcd}
C \arrow[r, "\beta"] \arrow[d, "\Delta"'] & A \\
C \otimes C \arrow[r, "\alpha"'] & A \otimes A \arrow[u, "m"']
\end{tikzcd}$$

So in $\mathrm{Hom}(C,A)$, the product is

$$\boxed{(f * g)(c) = \sum_{(c)} f(c_{(1)})\, g(c_{(2)})}$$

(u) We have $C \xrightarrow{\epsilon} \mathbb{K} \xrightarrow{u} A$ and we let this be the unit:

$$\boxed{1_{\mathrm{Hom}(C,A)} = u\epsilon}$$

Now let H = bialgebra

H^C = coalgebra of H

H^A = algebra of H

Then $\mathrm{Hom}(H^C, H^A)$ is an algebra

I (the identity) is an element of it

Prop $S : H \to H$ is an antipode for H if and only if $S * I = u\epsilon = I * S$ in $\mathrm{Hom}(H^C, H^A)$

$$u\epsilon \leftarrow 1$$

Pf $S * I(h) = \sum_{(c)} S(h_{(1)}) h_{(2)}$

$u(\epsilon(h)) = \epsilon(h) u(1) = \epsilon(h) 1$

$I * S(h) = \sum_{(c)} h_{(1)} S(h_{(2)})$ ∎

Corollary If S exists, it is unique

Pf Suppose S, S' are antipodes. Then

$$S = \overbrace{S * \underbrace{I * S'}_{1}}^{1} = S'.$$ ∎

Prop

in any Hopf algebra H

① $S(gh) = S(h) S(g)$ for all $g, h \in H$.

② $S \circ u = u$

③ $\epsilon \circ S = \epsilon$

④ $\Delta S(h) = \sum_{(h)} S(h_{(2)}) \otimes S(h_{(1)})$

①, ②: S is an algebra "antimorphism"

③, ④: S is a coalgebra "antimorphism".

Pf of ①: (②, ③, ④ similar)

Consider
$$M(g \otimes h) = gh \quad \text{in } \mathrm{Hom}((H \otimes H)^C, H^A)$$
$$N(g \otimes h) = S(h) S(g)$$
$$P(g \otimes h) = S(gh)$$

We claim $P * M = M * N = 1$.

As in groups, $(gh)^{-1} gh = (gh)(h^{-1} g^{-1}) = 1$

$$P * M(g \otimes h) = \sum_{(g \otimes h)} P((g \otimes h)_{(1)}) M((g \otimes h)_{(2)})$$

$\Delta \otimes \Delta(g \otimes h) = \Delta(g) \otimes \Delta(h)$
$\Rightarrow (g \otimes h)_{(1)} = g_{(1)} \otimes h_{(1)}$

$$= \sum_{(g),(h)} P(g_{(1)} \otimes h_{(1)}) M(g_{(2)} \otimes h_{(2)})$$

$$= \sum_{(g),(h)} S(g_{(1)} h_{(1)}) g_{(2)} h_{(2)}$$

$\Delta(gh) = \Delta(g) \Delta(h)$
$\Rightarrow (gh)_{(1)} = g_{(1)} h_{(1)}$

$$= \sum_{(gh)} S((gh)_{(1)}) (gh)_{(2)} = S * I(gh)$$

$$= \epsilon(gh) =$$

ϵ alg map

$$= \epsilon(g) \epsilon(h) = 1(g \otimes h) \quad \left(= ((u \otimes u)(\epsilon \otimes \epsilon))(g \otimes h)\right) \checkmark$$

$$M\otimes N(g\otimes h) = \sum_{(g\otimes h)} M((g\otimes h)_{(1)})\, N((g\otimes h)_{(2)})$$

$$= \sum_{(g)(h)} M(g_{(1)}\otimes h_{(1)})\, N(g_{(2)}\otimes h_{(2)})$$

$$= \sum_{(g)(h)} g_{(1)} h_{(1)} S(h_{(2)}) S(g_{(2)})$$

$I*S=1$

$$= \sum_{(g)} g_{(1)}\, \epsilon(h)\, S(g_{(2)})$$

$$= \sum_{(g)} g_{(1)} S(g_{(2)})\, \epsilon(h)$$

$I*S=1$

$$= \epsilon(g)\,\epsilon(h)$$

$$= 1(g\otimes h) \checkmark$$

So $P*M = M*N = 1$, and then

$P = P*M*N = N$ as desired. ∎

$\boxed{\text{Prop If } H \text{ is comm or cocomm, } S\circ S = I}$

Pf If H is comm, we have

$$u\epsilon(h) = S*I(h) = \sum_{(h)} S(h_{(1)}) h_{(2)}$$

$$= \sum_{(h)} h_{(2)} S(h_{(1)})$$

We claim $(S\circ S)*S = 1$, which together with $S*I=1$ gives $S\circ S = I$.

$$\big((S\circ S)*S\big)(h) = \sum_{(h)} (S\circ S)(h_{(1)})\, S(h_{(2)})$$

$$= \sum_{(h)} S\big(S(h_{(1)})\big)\, S(h_{(2)})$$

$S(gh) = S(h)S(g)$

$$= \sum_{(h)} S\big(h_{(2)} S(h_{(1)})\big)$$

$$= S\Big(\sum_{(h)} h_{(2)} S(h_{(1)})\Big)$$

$$= S u\epsilon(h)$$

$S\circ u = u$

$$= u\epsilon(h)$$

$$= 1(h)$$

∎

If H is cocomm,

$$u\epsilon(h) = I*S(h) = \sum_{(h)} h_{(1)} S(h_{(2)})$$

cocomm

$$= \sum_{(h)} h_{(2)} S(h_{(1)})$$

and proceed as above ∎

Prop. Sweedler's 4-D non-comm, non-cocomm bialgebra is a Hopf algebra and the antipode has order 4.

Pf

Need

$$\begin{tikzcd} & 1\otimes 1 \arrow[r] & 1\otimes S(1) \arrow[dr] & \\ 1 \arrow[ur] \arrow[r] \arrow[dr] & 1 \arrow[rr] & & 1 \\ & 1\otimes 1 \arrow[r] & S(1)\otimes 1 \arrow[ur] & \end{tikzcd}$$

$\boxed{S(1)=1}$

$$\begin{tikzcd} & g\otimes g \arrow[r] & g\otimes S(g) \arrow[dr] & \\ g \arrow[ur] \arrow[r] \arrow[dr] & 1 \arrow[rr] & & 1 \\ & g\otimes g \arrow[r] & S(g)\otimes g \arrow[ur] & \end{tikzcd}$$

- $gS(g)=1$
- $S(g)g=1$

$\boxed{S(g)=g}$

$$\begin{tikzcd} & x\otimes 1+g\otimes x \arrow[r] & x\otimes 1+g\otimes S(x) \arrow[dr] & \\ x \arrow[ur] \arrow[r] \arrow[dr] & 0 \arrow[rr] & & 0 \\ & x\otimes 1+g\otimes x \arrow[r] & S(x)\otimes 1+g\otimes x \arrow[ur] & \end{tikzcd}$$

- $x+gS(x)=0$
- $S(x)+gx=0$

$\boxed{S(x)=-gx}$

$$\begin{tikzcd} & gx\otimes g+1\otimes gx \arrow[r] & gx\otimes g+1\otimes S(gx) \arrow[dr] & \\ gx \arrow[ur] \arrow[r] \arrow[dr] & 0 \arrow[rr] & & 0 \\ & gx\otimes g+1\otimes gx \arrow[r] & S(gx)\otimes g+1\otimes gx \arrow[ur] & \end{tikzcd}$$

- $-x+S(gx)=0$
- $S(gx)g+gx=0$

$\boxed{S(gx)=x}$

So $S(1)=1$

$S(g)=g$

$S(x)=-gx \quad S^2(x)=-x \quad S^3(x)=gx \quad S^4(x)=x$

$S(gx)=x \quad S^2(gx)=-gx \quad S^3(gx)=-x \quad S^4(gx)=gx$

Prop S cannot have odd order

Pf If it did, say $S^{2k+1}=1$, we would have

$$S^{2k+1}(gh)=S^{2k+1}(h)S^{2k+1}(g)$$

$$gh=hg$$

so H would be commutative and S would have order 2

Prop. There exist Hopf algebras where S has order $2,4,6,8,\dots,$ and ∞.

Pf HW. (Taft, 1971).