What is a Hopf algebra?

Lecture 1.24.12

A $\Bbbk$-vector space with

$m: H \otimes H \to H$

("multiplication")

$$\begin{tikzcd}
H\otimes H\otimes H \arrow[r, "m\otimes id"] \arrow[d, "id\otimes m"'] & H\otimes H \arrow[d, "m"] \\
H\otimes H \arrow[r, "m"'] & H
\end{tikzcd}$$

associativity

$u: \Bbbk \to H$

("unit")

$$\begin{tikzcd}
 & H\otimes H \arrow[d, "m"] & \\
\Bbbk\otimes H \arrow[ur, "u\otimes id"] \arrow[r] & H & H\otimes \Bbbk \arrow[ul, "id\otimes u"'] \arrow[l]
\end{tikzcd}$$

unitary property

algebra structure

$\Delta: H \to H \otimes H$

("comultiplication")

$$\begin{tikzcd}
H\otimes H\otimes H & H\otimes H \arrow[l, "id\otimes\Delta"'] \\
H\otimes H \arrow[u, "\Delta\otimes id"] & H \arrow[l, "\Delta"] \arrow[u, "\Delta"']
\end{tikzcd}$$

coassociativity

$\epsilon: H \to \Bbbk$

("counit")

$$\begin{tikzcd}
 & H\otimes H \arrow[dl, "\epsilon\otimes id"'] \arrow[dr, "id\otimes\epsilon"] & \\
\Bbbk\otimes H & H \arrow[u, "\epsilon"'] \arrow[l] \arrow[r] & H\otimes \Bbbk
\end{tikzcd}$$

counitary property

coalgebra structure

$S: H \to H$

("antipode")

$$\begin{tikzcd}
 & H\otimes H \arrow[r, "S\otimes id"] & H\otimes H \arrow[dr, "m"] & \\
H \arrow[ur, "\Delta"] \arrow[r, "\varepsilon"] \arrow[dr, "\Delta"'] & \Bbbk \arrow[r, "u"] & & H \\
 & H\otimes H \arrow[r, "id\otimes S"'] & H\otimes H \arrow[ur, "m"'] &
\end{tikzcd}$$

antipode

(almidonado)

"We do not want to assume that the leadership is familiar with... Hopf algebra theory; and some find a direct passage to the starchy algebraist's diet too abrupt. This is why we start... with a motivational discussion." (Figueroa, Gracia-Bondía: Combinatorial Hopf algebras in quantum field theory.)

Intuitively, a Hopf algebra is a $\left(\begin{smallmatrix}\text{ring} \\ \text{vector space}\end{smallmatrix}\right)$ H with a <u>multiplication</u> $M: H\otimes H \to H$ and a <u>comultiplication</u> $\Delta: H \to H\otimes H$ which satisfy several restrictive rules.

Tensor Product of Vector Spaces

The <u>tensor product</u> $U\otimes V$ gives us a way of formally "multiplying" $u\in U, v\in V \mapsto$ "$u\otimes v$" $\in U\otimes V$.

$U\otimes V$ is $\Bbbk\text{-span}\{(u,v): u\in U, v\in V\} = F(U\times V)$

modulo the relations:

$$I \begin{cases} \bullet ((a+b), c) = (a, c) + (b, c) \\ \bullet (a, (b+c)) = (a, b) + (a, c) \\ \bullet ((\lambda a), b) = (a, (\lambda b)) = \lambda(a, b) \end{cases} \quad \lambda\in\Bbbk$$

If U, V are rings, $U\otimes V$ also with $(u\otimes v)(u'\otimes v') = uu'\otimes vv'$

Exercise: If $\{v_i\}_{i\in I}$, $\{v_j\}_{j\in J}$ are bases for U, V, then $\{v_i\otimes v_j\}_{i\in I, j\in J}$ is a basis for $U\otimes V$

<u>Ex</u>: $U = \mathbb{R}[x]$, $V = Mat_{2\times2}(\mathbb{R})$ are v.s. over $\mathbb{R}$

In $U\otimes V$ I can do

$$(2+2x)\otimes\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} = (1+x)\otimes\begin{pmatrix}0 & 2\\ -2 & 0\end{pmatrix}$$

$$\neq 2\otimes\begin{pmatrix}0 & 1+x\\ -1-x & 0\end{pmatrix}$$

<u>Ex</u>: $U = \Bbbk[x]$

An elt of $U\otimes U$ is

$$(1+2x)\otimes(x-2x^2) = (1\otimes x) + 2(x\otimes x) - 2(1\otimes x^2) - 4(x\otimes x^2)$$

⚠ • $(x\otimes x) \neq (1\otimes x^2)$

• Not every element is a "pure tensor" $u\otimes v$. (Example?)

Q What is $\Bbbk[x]\otimes\Bbbk[x]$?

<u>Examples of Hopf algebras</u>

Need • H – simultaneously a ring and vector space

• $m: H\otimes H \to H$

• $\Delta: H \to H\otimes H$

satisfying several properties (which we won't check yet)

1. <u>Groups</u>

G group $\to$ Group ring $\Bbbk G$

$\Bbbk$ field

$$\Bbbk G = \left\{ \sum_{i=1}^{n} \lambda_i g_i : \lambda_i \in \Bbbk, g_i \in G \right\}$$

Product: $m(g\otimes h) = gh$ (extend linearly)

Coproduct: $\Delta(g) = g\otimes g$ (extend linearly)

2. Polynomial Rings

$H = \Bbbk[x]$

$m(p(x)\otimes q(x)) = p(x)q(x)$

Product: $m(x^i \otimes x^j) = x^{i+j}$

Coproduct: $\Delta(x) = 1\otimes x + x \otimes 1$

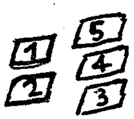
$$\begin{aligned}\Delta(x^2) &= \Delta(x\cdot x)\\ &= \Delta(x)\cdot\Delta(x)\\ &= (1\otimes x + x\otimes 1)\cdot(1\otimes x + x\otimes 1)\\ &= 1\otimes x^2 + 2x\otimes x + x^2\otimes 1\end{aligned}$$

$$\Delta(x^n) = \sum_{i=0}^{n} \binom{n}{i} x^i \otimes x^{n-i}$$

4. Permutations (Claudia Malvenuto 94)

$H = \Bbbk\{\text{permut's of } [n],\ n\geq 0\}$

product: "shuffling"



$12\otimes 321 = 12543 + 15243 + 15423 + 15432 + 51243 + 51423 + 51432 + 54123 + 54132 + 54312$

$U\otimes V$ = sum of shuffles of u and v

Coproduct: "cut and standardize"

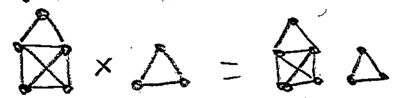
$\Delta(42531) = 1\otimes 42531 + 1\otimes 2431 + 21\otimes 321 + 213\otimes 21 + 3142\otimes 1 + 42531\otimes 1$

$$\Delta(u) = \sum_{i=0}^{n} st(u_1\ldots u_i)\otimes st(u_{i+1}\ldots u_n)$$

3. Graphs (Gian-Carlo Rota - late 70s, William Schmitt - late 80s)

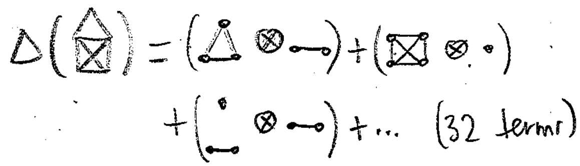
$H = \Bbbk\{\text{isomorphism classes of finite graphs}\}$

Product: disjoint union



Coproduct: cut into pieces

$$\Delta(G) = \sum_{S\subset V(G)} G|_S \otimes G|_{V-S}$$



Other important examples:

- Cohomology ring of a Lie group (Hopf, Samelson, Borel, 1940s)
- universal enveloping algebra of a Lie algebra
- quantum groups (Drinfeld 85)
- many more!

Ok, let's do this, one step at a time...

$\boxed{\text{Algebras}}$ ("associative algebras")

Idea: Simultaneously a ring and a vector space.

$\Bbbk$ – field $\quad$ A – ring with 1

Progressively more ~~complicated~~ flexible definitions:

(1.) A is a $\Bbbk$-algebra if $\underbrace{\Bbbk \subseteq Z(A)}$ and $1_{\Bbbk} = 1_A$

- $\Bbbk \subseteq A$
- $\lambda a = a\lambda$ for $\lambda \in \Bbbk$, $a \in A$

Exs:
- $A = \Bbbk[x]$
- $A = \Bbbk[x_1, \dots, x_n]$
- $A = \mathrm{Mat}_{n\times n}(\Bbbk)$?

(2.) A is a $\Bbbk$-algebra if there is a ring homom $u: \Bbbk \to A$ with $u(\Bbbk) \subseteq Z(A)$ and $u(1_{\Bbbk}) = 1_A$.

Ex: • $R = \mathrm{Mat}_{n\times n}(\Bbbk)$

$u: \Bbbk \to R$

$\lambda \mapsto \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \lambda I$

Observations:
- u must be injective, so $u(\Bbbk) \cong \Bbbk$
- A is a $\Bbbk$-vector space with $\lambda \cdot a := u(\lambda) a$
- R-algebras (R = ring with 1) are defined the same way. They are rings and R-modules
- $\mathbb{Z}$-algebra = ring

Homomorphisms:

$\phi: A_1 \to A_2$ is an algebra homom. if

$\phi(a+b) = \phi(a) + \phi(b)$
$\phi(ab) = \phi(a)\phi(b)$ } ring homom

$\phi(1_{A_1}) = 1_{A_2}$

$\phi(\lambda \cdot a) = \lambda \cdot \phi(a)$

→ vector space homom (linear map)

Observations:
- $B \subseteq A$ is a subalgebra if it is a subring and a subspace
- If I is an ideal of A, then A/I is naturally a quotient algebra. Isomorphism theorems hold:
- $A_1 \times A_2$ is naturally a direct product algebra. with $u: \Bbbk \to A_1 \times A_2$ given by $u = (u_1, u_2)$.
- $A_1 \otimes A_2$ is a tensor product. algebra.

(3.) A $\Bbbk$-algebra A is a $\Bbbk$-vector space with linear maps multiplication $m: A \otimes A \to A$ and unit $u: \Bbbk \to A$ such that these diagrams commute:

$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "m \otimes id"] \arrow[d, "id \otimes m"'] & A \otimes A \arrow[d, "m"] \\ A \otimes A \arrow[r, "m"'] & A \end{tikzcd}$$

$$\begin{tikzcd} & A \otimes A \arrow[d, "m"] & \\ \Bbbk \otimes A \arrow[ur, "u \otimes id"] \arrow[r] & A & A \otimes \Bbbk \arrow[ul, "id \otimes u"'] \arrow[l] \end{tikzcd}$$

associative:
$a(bc) = (ab)c$

Unitary:
$(a =)\, 1_k a = u(1_k) a$
$(a =)\, a 1_k = a u(1_k)$
$(\text{so } u(1_k) = 1_A)$

We have three defs of $\mathbb{k}$-algebra.

$1 \Rightarrow 2$

$2 \not\Rightarrow 1$, but $2 \Leftrightarrow 1$ if we identify $u(\mathbb{k})$ "=" $\mathbb{k}$

Why is 3 equivalent?

In 1,2, A is a vector space and a ring

In 3, it seems to only be a vector space

$3 \Rightarrow 2$

If A is a $\mathbb{k}$-alg in the sense of 3, I have

$$m: A \otimes A \to A$$

which I can use to define a product on A

$$\cdot : A \times A \mapsto A$$

by $\quad a \cdot b = m(a \otimes b)$

We also have $+$. Does this make A a ring?

$$\begin{aligned} \to \quad a(b+c) &= m(a \otimes (b+c)) \\ &= m(a\otimes b + a \otimes c) \\ &= m(a \otimes b) + m(a \otimes c) = ab + ac \end{aligned}$$

So $\cdot$ is distributive.

$\to$ Check other axioms similarly.

$2 \Rightarrow 3$

I have $\cdot : A \times A \to A$. Do I get $m: A \otimes A \to A$?

I do, since $\cdot$ is bilinear

Here we are using the characterization of $V \otimes W$ by a universal property. $\quad$ V, W, L vectorspace

Thm

- $\phi : V \times W \longrightarrow V \otimes W$ is bilinear
  $(v, w) \longmapsto v \otimes w$
  - $\to$ For fixed v, $\phi(v,w)$ linear in w
  - $\to$ For fixed w, $\phi(v,w)$ linear in v
- If $f : V \times W \to L$ is bilinear, then there is a unique linear map $\tilde{f} : V \otimes W \to L$ such that $f = \tilde{f} \circ \phi$

$$\begin{tikzcd} V \times W \arrow[r, "\phi"] \arrow[dr, "f"'] & V \otimes W \arrow[d, "\tilde{f}"] \\ & L \end{tikzcd}$$

So every bilinear map factors through ϕ; i.e., ϕ is the most general bilinear map from $V \times W$.

Proof

- ϕ is bilinear by the def of $V \otimes W$
- Given f bilinear, define

$$\bar{f} : F(V \times W) \to L \qquad F(V,W) = \mathbb{k}\{(v,w);\ v \in V, w \in W\}$$
$$(v,w) \mapsto f(v,w)$$

Now, $V \otimes W = F(V \times W)/I$ and $\bar{f}(I) = 0$ since f is bilinear, so this descends to the quotient

$$\tilde{f} : V \otimes W \to L$$

where $\tilde{f}(v \otimes w) = \bar{f}(v,w) = f(v,w)$.

Now we review tensor products:

$\boxed{\text{Lect. 4} \quad 2.2.12}$

$$V \otimes W = F(V \times W)/I$$

where

$$I = \langle (v+v', w) - (v,w) - (v',w),\ (v, w+w') - (v,w) - (v,w'),$$
$$(\lambda v, w) - \lambda(v,w),\ (v, \lambda w) - \lambda(v,w) \rangle$$

We write

$$v \otimes w = \overline{(v,w)} \text{ in } V \otimes W$$

- Keep in mind $v \otimes w$ a coset!
- It is tricky to tell whether $v \otimes w = v' \otimes w'$

To define a (vector space/$\mathbb{K}$-algebra) homomorphism

$$f : V \otimes W \to L$$

we can't define $f(v \otimes w)$ freely; we need to check, e.g., that $v \otimes w = v' \otimes w' \Rightarrow f(v \otimes w) = f(v' \otimes w')$

It is useful to just find

$$f : V \times W \to L$$

which is bilinear, and let it define $\tilde{f} : V \otimes W \to L$ by universality.

Ex 3 Do I have a "projection" $f : V \otimes W \to V$?

Look for a bilinear $f : V \times W \to V$

- $f(v,w) = v$ is not linear.
- $f(v,w) = v g(w)$ is linear if $g : W \to \mathbb{K}$ linear. get $f : V \otimes W \to V$ linear.

If $g : W \to \mathbb{K}$ is a $\mathbb{K}$-alg homom, so is f.

Ex 1 If A is a $\mathbb{K}$-algebra

$$\mathbb{K} \otimes A \cong A \text{ as } \mathbb{K}\text{-algebras}$$

- I need a homom. from $\mathbb{K} \otimes A$ to A. So take

$$f : \mathbb{K} \times A \to A$$
$$(\lambda, a) \mapsto \lambda a$$

which is bilinear, and descends to $f : \mathbb{K} \otimes A \to A$

- Check f is also a ring homom.: $f(\alpha) f(\beta) \overset{?}{=} f(\alpha\beta)$

Enough to check for pure tensors: $(\lambda a)(\lambda' a') = (\lambda\lambda')(aa')$

- Inverse: $g : A \to \mathbb{K} \otimes A$

$$a \mapsto 1 \otimes a$$

Ex 2 $\mathbb{C} \otimes \mathbb{R}[x] \cong \mathbb{C}[x]$ as $\mathbb{R}$-algebras

- Use $\mathbb{C} \times \mathbb{R}[x] \to \mathbb{C}[x]$; which is bilinear, to get
$(\lambda, p(x)) \mapsto \lambda p(x)$

Linear map $f : \mathbb{C} \otimes \mathbb{R}[x] \mapsto \mathbb{C}[x]$. Check $f(\alpha\beta) = f(\alpha) f(\beta)$ also

- The inverse is not so clear. Use 1st Isom Thm. Need:

a) f is surjective:

Any λx^n ($\lambda \in \mathbb{C}, n \in \mathbb{N}$) is $f(\lambda, x^n)$.

By linearity, get all of $\mathbb{C}[x]$

b) f is injective:

Sup $f(\sum_k \lambda_k \otimes p_k(x)) = 0$ $\qquad \lambda_k = a_k + i b_k$

$$\sum_k (a_k + i b_k) p_k(x) = 0$$

$$\sum_k a_k p_k(x) = 0 \qquad \sum_k b_k p_k(x) = 0$$

$$\Rightarrow \sum_k (\lambda_k \otimes p_k(x)) = \sum_k (a_k + i b_k) \otimes p_k(x)$$
$$= 1 \otimes \sum_k a_k p_k(x) + i \otimes \sum_k b_k p_k(x) = 0 \quad ■$$

Useful Lemma

If $\{w_j\}_{j\in J}$ is a basis for W and

$$\sum_i v_i \otimes w_i = 0 \quad \text{in } V\otimes W$$

then every $v_j = 0$

Pf Using $\{w_i\}_{i\in I}$ as a basis for W, use the linear $\pi_j : V\otimes W \longrightarrow W$ of Ex 3

with $g(w_i)=\begin{cases}1 & i=j\\ 0 & \text{otherwise}\end{cases}$

$$\pi_j(v_i\otimes w_i) = \begin{cases} 0 & i\neq j\\ v_j & i=j\end{cases}$$

Note: π_j is a homom of vector spaces, but not of $\mathbb{K}$-algebras.

I have

$$0=\pi_j\Big(\sum_i v_i\otimes w_i\Big) = v_j$$

for all j.

These examples should help you prove (in HW1) this key fact:

Prop If $\{v_i\}_{i\in I}, \{w_j\}_{j\in J}$ are bases for V, W, then $\{v_i\otimes w_j\}_{i\in I,\, j\in J}$ is a basis for $V\otimes W$.

Often people like to reason about (multi) linear algebra abstractly, without reference to a basis. For us, however, many of the Hopf algebras in combinatorics come equipped with natural bases.

Some (useful facts)/(instructive exercises):

- $U\otimes V \cong V\otimes U$
- $(U\otimes V)\otimes W \cong U\otimes (V\otimes W)$
- $(U\oplus V)\otimes W \cong (U\otimes W)\oplus (V\otimes W)$

as vec. sp. or $\mathbb{K}$-alg.

- To define $\left\{\begin{array}{ll} f: U\times V\to W & \text{bilinear}\\ \tilde{f}: U\otimes V \to W & \text{linear}\end{array}\right\}$ equivalent

we can choose bases $\{u_i\}_{i\in I}, \{v_j\}_{j\in J}$ of U, V,

- define $f(u_i, v_j)$ arbitrarily (in W)
- extend bilinearly

$$f(u,v) = f\Big(\sum_i \lambda_i u_i, \sum_j \mu_j v_j\Big) = \sum_{i,j}\lambda_i\mu_j f(u_i,v_j)$$

- (Extension of scalars)

If A is a $\mathbb{K}$-algebra, and $\mathbb{K}\subset\mathbb{L}$, then $A\otimes_{\mathbb{K}}\mathbb{L}$ is an $\mathbb{L}$-algebra.

(When we have ≥ 2 fields, we better keep track of which one we tensor over. Thus the $\otimes_{\mathbb{K}}$.)

Example: $\mathbb{R}[x]$ becomes a $\mathbb{C}$-algebra by $\otimes\mathbb{C}$;

we saw $\mathbb{R}[x]\otimes_{\mathbb{R}}\mathbb{C} \cong \mathbb{C}[x]$ as $\mathbb{R}$-algebras,

but also there is a natural action of $\mathbb{C}$

- From $\left\{\begin{array}{l}\text{vec. sp.}\\ \mathbb{K}\text{-alg}\end{array}\right\}$ maps $f_i : A_i \to B_i$ $(i=1,2)$, get $\left\{\begin{array}{l}\text{vec. sp.}\\ \mathbb{K}\text{-alg}\end{array}\right\}$ map $f_1\otimes f_2 : A_1\otimes A_2 \to B_1\otimes B_2$

Say A is <u>graded</u> if there is a decomposition

$$A = A_0 \oplus A_1 \oplus A_2 \oplus \cdots$$

such that $\underset{(\deg i)}{A_i}\,\underset{(\deg j)}{A_j} \subseteq \underset{(\deg i+j)}{A_{i+j}}$ for all i,j.

A_i: "homogeneous elts of degree i"

<u>Ex.</u>

- $A = \mathbb{K}[x] = (\mathbb{K}) \oplus (\mathbb{K}x) \oplus (\mathbb{K}x^2) \oplus \cdots$
- $A = \mathbb{K}[x_1, \ldots, x_n] = \bigoplus_{d=0}^{\infty} A_d$, where

  $A_d = \mathbb{K}\{x_1^{a_1} \cdots x_n^{a_n} : a_1 + \cdots + a_n = d,\ a_i \geq 0\}$

  $\binom{n+d-1}{d} = (-1)^d \binom{-n}{d}$
- $A = T(V) = \bigoplus_{d=0}^{\infty} V^{\otimes d}$ tensor algebra
- $A = S(V) = T(V) / \langle u \otimes v - v \otimes u : u, v \in V \rangle$ symmetric algebra
- $A = \Lambda(V) = T(V) / \langle v \otimes v : v \in V \rangle$ exterior algebra
- $A = \mathbb{K}\{\text{permuts. of } [n], \text{ some } n \in \mathbb{N}\}$

  $= \bigoplus_{n=0}^{\infty} \underset{\text{perms of } [n]}{\mathbb{K}S_n}$ recall: product = shuffles

General fact:

If A is graded, an ideal I is <u>homogeneous</u> if it is generated by homogeneous elements.
If so, A/I is graded.

The <u>Hilbert series</u> of A is $\mathrm{Hilb}(A;q) = \sum_{d=0}^{\infty} (\dim A_d)\, q^d$.

We will say much more about algebras, but let us now introduce

$\boxed{\text{Coalgebras}}$

A <u>coalgebra</u> is a $\mathbb{K}$-vector space C with linear maps, $\underset{\text{(comultiplication)}}{\Delta : C \to C \otimes C}$, $\underset{\text{(counit)}}{\epsilon : C \to \mathbb{K}}$

such that

$$\begin{tikzcd}
C \otimes C \otimes C & C \otimes C \arrow[l, "id \otimes \Delta"'] \\
C \otimes C \arrow[u, "\Delta \otimes id"] & C \arrow[l, "\Delta"'] \arrow[u, "\Delta"']
\end{tikzcd}$$

coassociativity

and

$$\begin{tikzcd}
 & C \otimes C \arrow[dl, "\epsilon \otimes id"'] \arrow[dr, "id \otimes \epsilon"] & \\
\mathbb{K} \otimes C & & C \otimes \mathbb{K} \\
 & C \arrow[ul, "\cong"] \arrow[uu, "\Delta"] \arrow[ur, "\cong"'] &
\end{tikzcd}$$

counitary prop.

commute

(Same diagrams as with algebras, but all arrows reversed.)

<u>Ex 1</u>: S = set $\quad \mathbb{K}S$ = vec. sp. with S as basis.

$\begin{cases} \Delta(s) = s \otimes s, & \text{extend linearly} \\ \epsilon(s) = 1, & \text{"} \quad \text{"} \end{cases}$ (Check!)

(We did this for S = group, but we get a coalgebra for any set S)

Ex 2: Incidence Coalgebra

Let P = poset

For $x \leq y$, the <u>interval</u> $[x,y] = \{z \in P : x \leq z \leq y\}$

$Int(P) := \{\text{intervals in } P\} = \{[x,y] : x \leq y \text{ in } P\}$

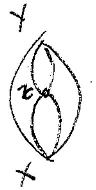
$$C = \mathbb{K}\,Int(P)$$

Define

$$\Delta([x,y]) = \sum_{z : x \leq z \leq y} [x,z] \otimes [z,y]$$

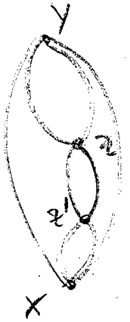
$$\epsilon([x,y]) = \begin{cases} 1 & x = y \\ 0 & x < y \end{cases}$$

and extend linearly.



Check coassociativity:

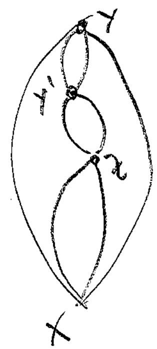
- $(\Delta \otimes id)\left(\sum_{z : x \leq z \leq y} [x,z] \otimes [z,y]\right) =$



$$= \sum_{z : x \leq z \leq y} \left(\sum_{z' : x \leq z' \leq z} [x,z'] \otimes [z',z] \right) \otimes [z,y]$$

$$= \sum_{x \leq z' \leq z \leq y} [x,z'] \otimes [z',z] \otimes [z,y]$$

- $(id \otimes \Delta)\left(\qquad \qquad \right) =$



$$= \sum_{x \leq z \leq y' \leq y} [x,z] \otimes [z,y'] \otimes [y',y]$$

Check counitary property.

You may think algebras are more natural than coalgebras, but they are more or less equivalent:

Duality

If V is a $\mathbb{K}$-vector space, $V^* = \mathrm{Hom}_{\mathbb{K}}(V, \mathbb{K})$ is the dual vector space

linear "functionals" from V to $\mathbb{K}$

($V \cong V^*$ but it is useful to remember which is which)

"vectors" "functionals"

It is called duality because $(V^*)^* = V$.

Fact.
If C is a coalgebra, C^* is naturally an algebra.
If A is a <u>fin. dim.</u> algebra, A^* is naturally a coalgebra.

To see this, note/recall

- I have $\langle , \rangle : V^* \times V \to \mathbb{K}$ bilinear.

  $\langle v^*, v \rangle := v^*(v)$

- I have $\rho : V^* \otimes W^* \to (V \otimes W)^*$, where $\rho(v^* \otimes w^*)$ (for $v^* \in V^*, w^* \in W^*$) is given by

  $$\langle \rho(v^* \otimes w^*), v \otimes w \rangle = \langle v^*, v \rangle \langle w^*, w \rangle$$

  - This is always injective. (Prove it.)
  - If V, W are fin-dim, this is bijective. (Ex.)

- If $L : V \to W$ is linear, we get $L^* : W^* \to V^*$.

For $w^* \in W^*$, $L^* w^* \in V^*$ is $\langle L^* w^*, v \rangle = \langle w^*, Lv \rangle$.

$\boxed{\begin{array}{ll} \text{Let } & C \text{ be a coalgebra with} \\ \text{comult} & \Delta : C \to C \otimes C \\ \text{counit} & \varepsilon : C \to \mathbb{K} \end{array}}$

Let $A = C^*$ with

mult
$$\begin{tikzcd} C^* \otimes C^* \arrow[rr, "m"] \arrow[dr, "\rho"'] & & C^* \\ & (C \otimes C)^* \arrow[ur, "\Delta^*"'] & \end{tikzcd}$$

unit
$$\begin{tikzcd} \mathbb{K} \arrow[rr, "u"] \arrow[dr, "\cong"'] & & C^* \\ & \mathbb{K}^* \arrow[ur, "\varepsilon^*"'] & \end{tikzcd}$$

where we use $f : \mathbb{K}^* \xrightarrow{\cong} \mathbb{K}$

$$v \mapsto v(1)$$

$\boxed{\underline{\text{Prop}} \ (C^*, m, u) \text{ is an algebra}}$

<u>Pf</u> Homework.

Similarly

$\boxed{\begin{array}{ll} \text{Let } & A \text{ be an algebra with} \\ \text{mult:} & m : A \otimes A \to A \\ \text{unit:} & u : \mathbb{K} \to A \end{array}}$

$\boxed{\text{If } A \text{ is fin.-dim}}$, a similar construction gives $\Delta : A^* \to A^* \otimes A^*$, $\epsilon : A^* \to \mathbb{K}$ and

$\boxed{(A^*, \Delta, \epsilon) \text{ is a coalgebra.}}$

<u>Example</u>: (Very important in enumerative combinatorics)

Let $C(P)$ = incidence coalgebra of P

Dual: $\boxed{A(P) = \text{incidence algebra of } P}$

<u>Elts of A</u>: Linear functionals $c^* : C \to \mathbb{K}$

$$\Updownarrow$$

functions $\boxed{c^* : \mathrm{Int}(P) \to \mathbb{K}}$

<u>Multiplication</u>:
$$\begin{array}{ccc} m : A \otimes A & \longrightarrow & A \\ \shortparallel & & \shortparallel \\ C^* \otimes C^* & & C^* \end{array}$$

$$c^* \otimes d^* \longmapsto c^* d^*$$

is given by

$$\begin{aligned} c^* d^*([x,y]) &= \langle m(c^* \otimes d^*), [x,y] \rangle \\ &= \langle \Delta^* \rho(c^* \otimes d^*), [x,y] \rangle \\ &= \langle \rho(c^* \otimes d^*), \Delta [x,y] \rangle \\ &= \langle \rho(c^* \otimes d^*), \sum_{x \le z \le y} [x,z] \otimes [z,y] \rangle \end{aligned}$$

$$\boxed{c^* d^*([x,y]) = \sum_{x \le z \le y} c^*[x,z] \, d^*[z,y]}$$

Convolution

<u>Unit</u>: $u(1) \in C^*$ $\langle u(1), [x,y] \rangle = \langle \varepsilon^* 1^*, [x,y] \rangle = \langle 1^*, \varepsilon[x,y] \rangle$
$$= \begin{cases} 1 & x = y \\ 0 & x < y \end{cases}$$

$$\boxed{\delta[x,y] = \begin{cases} 1 & x = y \\ 0 & x < y \end{cases}}$$

Sweedler notation for coalgebras

In a coalgebra C we have

$$\Delta : C \to C \otimes C$$

$$\Delta(c) = \sum_{i=1}^{n} c_{1i} \otimes c_{2i} \qquad (c, c_{1i}, c_{2i} \in C)$$

We write

variable in C

$$\boxed{\Delta(c) = \sum_{(c)} c_{(1)} \otimes c_{(2)}}$$

who did you take the coproduct of?

(You may not like this at first; it's like writing $\sum_{i=1}^{n} a_i$ as $\sum a$. In fact, some people even write $\Delta(c) = c_{(1)} \otimes c_{(2)}$, which is like writing $\sum_{i=1}^{n} a_i$ as a.
But this is very useful notation once you get used to it!)

Similarly, in

$$\begin{tikzcd} C\otimes C\otimes C & C\otimes C \arrow[l] \\ C\otimes C \arrow[u] & C \arrow[l] \arrow[u] \end{tikzcd}$$

, write

$$\Delta_2(c) = \sum_{(c)} c_{(1)} \otimes c_{(2)} \otimes c_{(3)}$$

for

$$(\Delta \otimes I)\Delta(c) = (I \otimes \Delta)\Delta(c)$$

and more generally

$$\boxed{\Delta_{n-1}(c) = \sum_{(c)} c_{(1)} \otimes \cdots \otimes c_{(n)}}$$

Also, any multilinear $f : C \times C \times \cdots \times C \to V$ gives a linear $\bar{f} : C \otimes C \otimes \cdots \otimes C \to V$ and we write

$$\boxed{\bar{f}\Delta_{n-1}(c) = \sum_{c} f(c_{(1)}, c_{(2)}, \ldots, c_{(n)})}$$

Ex $\sum_{(c)} \Delta(c_{(1)}) \otimes c_{(2)}$ (comes from $f(c,d) = \Delta(c) \otimes d = (\Delta \otimes I)(c \otimes d)$)

$$= \sum_{(c)} (\Delta \otimes I)(c_{(1)} \otimes c_{(2)}) = (\Delta \otimes I)\Delta(c) = \Delta_2(c)$$

So we can write coassociativity in Sweedler notation as

$$\sum_{(c)} \Delta(c_{(1)}) \otimes c_{(2)} = \sum_{(c)} c_{(1)} \otimes \Delta(c_{(2)})$$

Ex The counitary property is (check it)

$$\sum_{(c)} \epsilon(c_{(1)}) \otimes c_{(2)} = \sum_{(c)} c_{(1)} \otimes \epsilon(c_{(2)}) = c$$

Ex Prove

$$\begin{tikzcd} & C\otimes C\otimes C \arrow[dl] & C\otimes C \arrow[l] \\ C\otimes k\otimes C \arrow[dr] & & \\ & C\otimes C \arrow[uu] & C \arrow[l] \arrow[uu] \end{tikzcd}$$

$$\sum_{(c)} c_{(1)} \otimes \epsilon(c_{(2)}) \otimes c_{(3)} = \sum_{(c)} c_{(1)} \otimes c_{(2)}$$

Homomorphisms

Lect 7
2.13.12

Homomorphisms of algebras: (revisited)

A, B $\mathbb{K}$-algebras

$f : A \to B$ linear

Claim:

f is a $\mathbb{K}$-algebra homomorphism

$$\Updownarrow$$

$$\begin{tikzcd} A \otimes A \arrow[r, "f \otimes f"] \arrow[d, "m_A"'] & B \otimes B \arrow[d, "m_B"] \\ A \arrow[r, "f"'] & B \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} A \arrow[rr, "f"] & & B \\ & \mathbb{K} \arrow[ul, "u_A"] \arrow[ur, "u_B"'] & \end{tikzcd}$$

commute

Pf: f is a homom. of vector spaces. Ring structure:

Left diagram says
$f(a_1 a_2) = f(a_1) f(a_2)$
for all $a_1, a_2 \in A$

Right diagram says
$f(1_A) = f(1_B)$

$\blacksquare$

B $\mathbb{K}$-algebra
$A \subseteq B$ subspace

- A is a subalgebra if $m(A \otimes A) \subset A$, $1_B \in A$.
- A is an ideal (two-sided) if $m(A \otimes B) \subset A$, $m(B \otimes A) \subset A$

Homomorphisms of coalgebras:

C, D $\mathbb{K}$-coalgebras

Def.

A linear $g : C \to D$ is a $\mathbb{K}$-coalgebra homom. if

$$\begin{tikzcd} C \otimes C \arrow[r, "g \otimes g"] & D \otimes D \\ C \arrow[u, "\Delta_C"] \arrow[r, "g"'] & D \arrow[u, "\Delta_D"'] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} C \arrow[rr, "g"] \arrow[dr, "\epsilon_C"'] & & D \arrow[dl, "\epsilon_D"] \\ & \mathbb{K} & \end{tikzcd}$$

commute

In Sweedler notation,

- $\sum_{(c)} g(c_{(1)}) \otimes g(c_{(2)}) = \sum_{(g(c))} (g(c))_{(1)} \otimes (g(c))_{(2)}$

and

- $\epsilon_C(c) = \epsilon_D(g(c))$

D $\mathbb{K}$-coalgebra
$C \subseteq D$ subspace

- C is a subcoalgebra if $\Delta(C) \subset C \otimes C$
- C is a coideal (two-sided) if $\Delta(C) \subset C \otimes D + D \otimes C$

Duality of homomorphisms:

Prop If $f: C \to D$ is a coalgebra map then $f^*: D^* \to C^*$ is an algebra map.

Pf. We check $f^*(d^*e^*) = f^*(d^*)f^*(e^*)$ in C^*.

$$\begin{aligned}\langle f^*(d^*e^*), c\rangle &= \langle d^*e^*, f(c)\rangle \\ &= \langle m(d^*\otimes e^*), f(c)\rangle \quad m = \underset{\text{inclusion}}{\varphi}\Delta^* \text{"="} \Delta^* \\ &= \langle d^*\otimes e^*, \Delta f(c)\rangle \quad f \text{ homom} \\ &= \langle d^*\otimes e^*, (f\otimes f)\Delta(c)\rangle \\ &= \langle d^*\otimes e^*, (f\otimes f)\sum_{(c)} c_{(1)}\otimes c_{(2)}\rangle \\ &= \sum_{(c)} \langle d^*, f(c_{(1)})\rangle\langle e^*, f(c_{(2)})\rangle \\ &= \sum_{(c)} \langle f^*(d^*), c_{(1)}\rangle\langle f^*(e^*), c_{(2)}\rangle\end{aligned}$$

$$\begin{aligned}\langle f^*(d^*)f^*(e^*), c\rangle &= \langle m(f^*(d^*)\otimes f^*(e^*)), c\rangle \\ &= \langle f^*(d^*)\otimes f^*(e^*), \Delta c\rangle \\ &= \langle f^*(d^*)\otimes f^*(e^*), \sum_{(c)} c_{(1)}\otimes c_{(2)}\rangle \\ &= \sum_{(c)} \langle f^*(d^*), c_{(1)}\rangle\langle f^*(e^*), c_{(2)}\rangle\end{aligned}$$

Also need $f^*(1_{D^*}) = 1_{C^*}$ (Exercise)

Similarly,:

Prop If $f: A \to B$ is a map of fin. dim. algebras, then $f^*: B^* \to A^*$ is a map of coalgebras.

Review: Dual subspaces

V vector space

V^* dual

Given $\underset{\text{subspace}}{S \subset V}$, let $S^\perp := \{v^* \in V^* : \langle v^*, S\rangle = 0\} \underset{\text{subspace}}{\subset} V^*$

Given $T^* \subset V^*$, let $(T^*)^\perp = \{v \in V : \langle T^*, v\rangle = 0\} \subset V$.

Prop Let C be a coalgebra, C^* the dual algebra. Let $D \subset C$ be a subspace.

D is a coalgebra of C

$\Updownarrow$

$D^\perp$ is an ideal of C^*

Recall: $(S^\perp)^\perp = S$, $(T^{*\perp})^\perp = T^*$

Pf: $\Downarrow$. The inclusion $i: D \hookrightarrow C$ gives an algebra map $i^*: C^* \to D^*$.

$c^* \in \mathrm{Ker}(i^*) \iff i^*(d) = 0$ for all $d \in D$

$\iff i^* \in D^\perp$

So $D^\perp = \mathrm{Ker}(i^*)$ is an ideal.

⇑: Let $D^\perp$ be an ideal of C^*, and $x \in D$.

We need $\Delta(x) \in D \otimes D$.

Let $\Delta(x) = \sum_i y_i \otimes z_i$

Note: We can assume $\{z_i\}$ is lin. indep.; if $z_j = \sum_k \alpha_k z_k$, replace $y_j \otimes z_j$ by $\sum_k \alpha_k y_j \otimes z_k$ and absorb it into the other tensors.

Claim: $y_j \in D$ for any j.

Assume $y_j \notin D$.

Then find $d^* \in D^\perp$ so $\langle y_j, d^* \rangle \neq 0$

Also find $z^* \in C^*$ so $\langle z_j, z^* \rangle = 1$, $\langle z_i, z^* \rangle = 0$ $(i \neq j)$

$\underline{z^* \in C^*}$: $d^* z^* \in D^\perp$ since $D^\perp$ is ideal

Then

$$0 = \langle x, d^* z^* \rangle = \langle x, m(d^* \otimes z^*) \rangle$$

$$= \langle \Delta(x), d^* \otimes z^* \rangle = \sum_i \langle y_i, d^* \rangle \langle z_i, z^* \rangle$$

$$= \langle y_j, d^* \rangle \neq 0 \quad ✓$$

Therefore $\Delta(x) \in D \otimes C$.

Similarly $\Delta(x) \in C \otimes D$

But $(D \otimes C) \cap (C \otimes D) = D \otimes D$. ∎

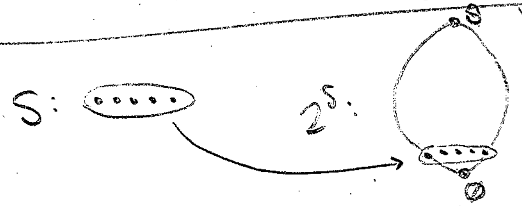
↑ Exercise.

An example of a coalgebra homomorphism:

$$S \text{ set} \to C = \mathbb{K}S \text{ coalgebra}$$

$$S \text{ set} \to 2^S = \{T : T \subseteq S\} \text{ poset} \to C(2^S) \text{ incidence coalgebra}$$

$$\boxed{\begin{array}{l} \text{Let } f : \mathbb{K}S \longrightarrow C(2^S) \\ \qquad s \longmapsto [\{s\},\{s\}] \quad \text{for } s \in S \end{array}}$$



write $[s,s]$

Coassoc:

$$\begin{tikzcd} \mathbb{K}S \otimes \mathbb{K}S \arrow[r, "f \otimes f"] & C(2^S) \otimes C(2^S) \\ \mathbb{K}S \arrow[u, "\Delta"] \arrow[r, "f"'] \arrow[ur, phantom, "?"] & C(2^S) \arrow[u, "\Delta"'] \end{tikzcd}$$

In $C(2^S)$,
$$\Delta([T,T]) = \sum_{T \subseteq U \subseteq T} [T,U] \otimes [U,T] = [T,T] \otimes [T,T]$$
for all $T \subseteq S$

Indeed:

$$\begin{tikzcd} s \otimes s \arrow[r, mapsto] & {[s,s] \otimes [s,s]} \\ s \arrow[u] \arrow[r, mapsto] & {[s,s]} \arrow[u] \end{tikzcd}$$

Counit:

$$\begin{tikzcd} \mathbb{K}S \arrow[r, "f"] \arrow[dr] & C(2^S) \arrow[d] \\ & \mathbb{K} \end{tikzcd} \quad ?$$

Indeed:

$$\begin{tikzcd} s \arrow[r, mapsto] \arrow[dr] & {[s,s]} \arrow[d] \\ & 1 \end{tikzcd}$$

Dually,

Lect 8
2.15.12

Prop Let C be a coalgebra
C^* the dual algebra
$V \subset C$ be a subspace
Then $V \subseteq C$ is a coideal
$\Updownarrow$
$V^\perp \subseteq C^*$ is a subalgebra

Our next goal: Fundamental ("First") Isomorphism Theorem for coalgebras.

First:

Lemma: Let $f: V \to V'$, $g: W \to W'$ be linear. Then we get $f \otimes g: V \otimes W \to V' \otimes W'$, and

(a) $\mathrm{Im}(f \otimes g) = \mathrm{Im} f \otimes \mathrm{Im}\, g$

(b) $\mathrm{Ker}(f \otimes g) = \mathrm{Ker} f \otimes W + V \otimes \mathrm{Ker}\, g$

(a) is straightforward

(b) $\supseteq$ A generator of the right side is
$a \otimes w + v \otimes b$ $\quad f(a) = 0 \quad g(b) = 0$

and
$$(f \otimes g)(a \otimes w + v \otimes b) = 0 \otimes g(w) + f(v) \otimes 0 = 0$$

$\subseteq$ Let $U = \text{RHS} \subset \mathrm{Ker}(f \otimes g)$

Then $f \otimes g: V \otimes W \to V' \otimes W'$ descends to $\overline{f \otimes g}: (V \otimes W)/U \to \mathrm{Im} f \otimes \mathrm{Im}\, g \ (\subset V' \otimes W')$

To show this is an isom, we build an inverse.

Let $\phi: \mathrm{Im} f \otimes \mathrm{Im}\, g \to (V \otimes W)/U$
$$f(v) \otimes g(w) \mapsto v \otimes w + U$$

Well-defined:

If $f(v_1) \otimes g(w_1) = f(v_2) \otimes g(w_2)$ then $f(v_1) = f(v_2)$ and $g(w_1) = g(w_2)$, so $v_1 - v_2 \in \mathrm{Ker} f$, $w_1 - w_2 \in \mathrm{Ker}\, g$, and
$$v_1 \otimes w_1 - v_2 \otimes w_2 = v_1 \otimes (w_1 - w_2) + (v_1 - v_2) \otimes w_2 \in V \otimes \mathrm{Ker}\, g + \mathrm{Ker} f \otimes W$$

Really defined:

$\mathrm{Im} f \times \mathrm{Im}\, g \to (V \otimes W)/U$
$(f(v), g(w)) \mapsto v \otimes w + U$ is bilinear

Now, $\overline{f \otimes g}$ and ϕ are clearly inverses on the generators, so they are inverses.

We conclude $\overline{f \otimes g}$ is injective, so $\mathrm{Ker}(f \otimes g) = U$. (If $\alpha \in \mathrm{Ker}(f \otimes g)$, $\alpha + U \in \mathrm{Ker}(\overline{f \otimes g}) = 0 + U$)

With that, ...

Let $f: C \to D$ be a coalgebra map

Prop $\ker f$ is a coideal of C

Pf If $c \in \ker f$ then $f(c) = 0$ gives

$$0 = \Delta f(c) = (f \otimes f)\Delta(c)$$

since f is a coalgebra map.

But then

$$\Delta(c) \in \ker(f \otimes f) = \ker f \otimes C + C \otimes \ker f \qquad \blacksquare$$

Prop $\operatorname{Im} f$ is a subcoalgebra of D

Pf Let $f(c) \in \operatorname{Im} f$.

Then
$$\Delta f(c) = (f \otimes f)\Delta(c) = \sum_{(c)} f(c_{(1)}) \otimes f(c_{(2)}) \in \operatorname{Im} f \otimes \operatorname{Im} f \qquad \blacksquare$$

Prop If I is a coideal of C, then C/I inherits the coalgebra structure from C ← quotient coalgebra

Pf Need $C/I \to C/I \otimes C/I$

Have:

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d] & C \otimes C \arrow[d, "\pi \otimes \pi"] \\ C/I \arrow[r, dotted, "?"] & C/I \otimes C/I \end{tikzcd}$$

The map $(\pi \otimes \pi) \circ \Delta : C \to C/I \otimes C/I$ descends to C/I iff it sends I to 0.

But it does: for $i \in I$

$$(\pi \otimes \pi) \underbrace{\Delta(i)}_{\in I \otimes C + C \otimes I} = 0 \qquad \blacksquare$$

Fundamental Theorem of Coalgebras

If $f: C \to D$ is a map of coalgebras, then $\operatorname{Im} f \cong C/\ker f$ as coalgebras

Pf Straightforward. $\blacksquare$

Other constructions:

If C and D are coalgebras,

- The <u>tensor product coalgebra</u> has coproduct

$$C\otimes D \xrightarrow{\Delta_C\otimes\Delta_D} C\otimes C\otimes D\otimes D \xrightarrow{I\otimes T\otimes I} C\otimes D\otimes C\otimes D$$

$$(T(c\otimes d) = d\otimes c)$$

and counit

$$C\otimes D \xrightarrow{\epsilon_C\otimes\epsilon_D} \mathbb{K}\otimes\mathbb{K} \longrightarrow \mathbb{K}$$

- The <u>direct sum coalgebra</u> has coproduct

$$C\oplus D \xrightarrow{(\Delta_C,\Delta_D)} (C\otimes C)\oplus(D\otimes D) \hookrightarrow (C\oplus D)\otimes(C\oplus D)$$

Bialgebras

Suppose (H,m,u) is an algebra
(H,Δ,ϵ) is a coalgebra.

<u>Def/Thm</u> We say (H,m,u,Δ,ϵ) is a bialgebra if the following equivalent conditions hold:

1. m and u are coalgebra maps
2. Δ and ϵ are algebra maps
3. - $\Delta(1) = 1\otimes 1$
   - $\Delta(gh) = \sum_{(g)(h)} g_{(1)}h_{(1)}\otimes g_{(2)}h_{(2)}$
   - $\epsilon(1) = 1$ and
   - $\epsilon(gh) = \epsilon(g)\epsilon(h)$

② $\Leftrightarrow$ ③ is clean. Each of ① and ② gives 4 comm. dgms. <u>Key</u>: m respects Δ $\Leftrightarrow$ Δ respects m

$$\begin{tikzcd}
H\otimes H \arrow[r, "m"] \arrow[d, "\Delta\otimes\Delta"'] \arrow[drr, bend right, "\Delta_{H\otimes H}"'] & H \arrow[r, "\Delta"] \arrow[d, phantom, "\text{Commutes}"] & H\otimes H \\
H\otimes H\otimes H\otimes H \arrow[rr, "I\otimes T\otimes I"] \arrow[urr, bend right, "m_{H\otimes H}"'] & & H\otimes H\otimes H\otimes H \arrow[u, "m"']
\end{tikzcd}$$

Examples of bialgebras

1. Group ring

$H = \mathbb{K}G \qquad G$ group

$$\begin{tikzcd} g\otimes h \arrow[r, "m"] \arrow[d, "\Delta\otimes\Delta"'] & gh \arrow[r, "\Delta"] & gh\otimes gh \\ g\otimes g\otimes h\otimes h \arrow[rr, "I\otimes T\otimes I"] & & g\otimes h\otimes g\otimes h \arrow[u, "m\otimes m"'] \end{tikzcd}$$

Check three other diagrams.

2. Polynomial ring

$H = \mathbb{K}[x]$ mult, unit: usual counit: $\epsilon(x^n) = \begin{cases} 1 & n=0 \\ 0 & n\geq 1 \end{cases}$ extend

comult: $\Delta(x) = x\otimes 1 + 1\otimes x$, multiplicatively

$$\Rightarrow \Delta(x^n) = \sum_{k=0}^{n} \binom{n}{k} x^k \otimes x^{n-k}$$

$$\begin{tikzcd} x^a\otimes x^b \arrow[r, "m"] \arrow[d] & x^{a+b} \arrow[r, "\Delta"] & \sum_{k=0}^{a+b} \binom{a+b}{k} x^k \otimes x^{a+b-k} \\ \left(\sum_{i=0}^{a} \binom{a}{i} x^i \otimes x^{a-i}\right)\left(\sum_{j=0}^{b} \binom{b}{j} x^j \otimes x^{b-j}\right) \arrow[rr] & & \sum \binom{a}{i}\binom{b}{j} x^i \otimes x^j \otimes x^{a-i} \otimes x^{b-j} \arrow[u] \end{tikzcd}$$

So $\binom{a+b}{k} = \sum_{i+j=k} \binom{a}{i}\binom{b}{j}$

3. Posets

Let $\mathcal{I} = \mathbb{K}\{$isomorphism classes of posets with $\hat{0}$ and $\hat{1}\}$

Notation abuse: $\overline{P}$ = isom. class of P. Write P for $\overline{P}$.

Coalgebra:

$$\Delta(P) = \sum_{p\in P} [\hat{0}, p] \otimes [p, \hat{1}]$$

$$\epsilon(P) = \begin{cases} 1 & \text{if } \overline{P} = \cdot \\ 0 & \text{otherwise} \end{cases}$$

Algebra:

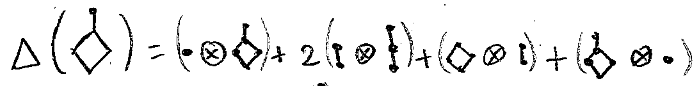
$m(P\otimes Q) = P\times Q =: \text{"}P.Q\text{"}$

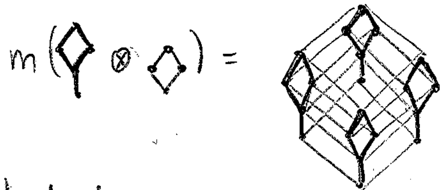
$u(1) = \overline{\cdot}$

Poset product:

$P\times Q = \{(p,q) : p\in P, q\in Q\}$

$(p,q) \leq (p',q') \iff p\leq p', q\leq q'$

Ex: 



Bialgebra:

$$\Delta(P\times Q) = \sum_{(p,q)\in P\times Q} [(0,0),(p,q)] \otimes [(p,q),(1,1)]$$

$$= \sum_{p\in P, q\in Q} ([0,p]\times[0,q]) \otimes ([p,1]\times[q,1])$$

$$= \sum_{p\in P, q\in Q} ([0,p]\otimes[p,1]) \times ([0,q]\otimes[q,1]) = \Delta(P)\times\Delta(Q)$$

m on $A\otimes B$: $m(a\otimes b, a'\otimes b') := m(a,a')\otimes m(b,b')$

$\epsilon(P\times Q) = \epsilon(P)\,\epsilon(Q)$ (Check)

Def. A coalgebra C is cocommutative if

$$\begin{tikzcd} & C\otimes C \arrow[dd, "T"] \\ C \arrow[ur, "\Delta"] \arrow[dr, "\Delta"'] & \\ & C\otimes C \end{tikzcd}$$

commutes

In Sweedler notation,

$$\Delta(c) = \sum_{(c)} c_{(1)} \otimes c_{(2)} = \sum_{(c)} c_{(2)} \otimes c_{(1)} \quad (*)$$

An element satisfying $(*)$ is a cocommutative elt.

Note: If C is spanned by cocomm. elts, it is cocomm.

	comm	cocomm
Ex 1	no	yes
Ex 2	yes	yes
Ex 3	yes	no
HW 2.1	no	no

Ex 4 (gen. of Ex 1)

Let G be a monoid: (a set with a binary operation which has a 1 and is associative)

(Think: "group without inverses")

Example: $G = 2^S$, $A\cdot B = A \cap B$ for $A, B \subseteq S$

Then the monoid algebra $\mathbb{K}G$ with

$m(g,h) = g\cdot h \quad u(1) = 1_G \quad \Delta(g) = g\otimes g \quad \epsilon(g) = 1$

is a bialgebra

Hopf algebras

Def A Hopf algebra is a bialgebra H with a linear map $S: H \to H$, called the antipode, such that this diagram commutes:

$$\begin{tikzcd} & H\otimes H \arrow[rr, "S\otimes I"] & & H\otimes H \arrow[dr, "m"] & \\ H \arrow[ur, "\Delta"] \arrow[rr, "\epsilon"] \arrow[dr, "\Delta"'] & & \mathbb{K} \arrow[rr, "u"] & & H \\ & H\otimes H \arrow[rr, "I\otimes S"] & & H\otimes H \arrow[ur, "m"'] & \end{tikzcd}$$

In Sweedler notation, S should satisfy

$$\boxed{\sum_{(h)} h_{(1)} S(h_{(2)}) = \epsilon(h)1 = \sum_{(h)} S(h_{(1)}) h_{(2)}}$$

Idea: S is some kind of analog to an inverse.

Ex 1. $H = \mathbb{K}G$ group ring

We saw it is a bialgebra.

Claim: $S(g) = g^{-1}$ makes it a Hopf algebra:

$$\begin{tikzcd} & g\otimes g \arrow[r, mapsto] & g^{-1}\otimes g \arrow[dr, mapsto] & \\ g \arrow[ur, mapsto] \arrow[r, mapsto] \arrow[dr, mapsto] & 1 \arrow[rr, mapsto] & & 1 \\ & g\otimes g \arrow[r, mapsto] & g\otimes g^{-1} \arrow[ur, mapsto] & \end{tikzcd}$$