(5) Nice Posets (in collaboration with Brian Cruz, Karla Lanzas, and Crista Moreno)

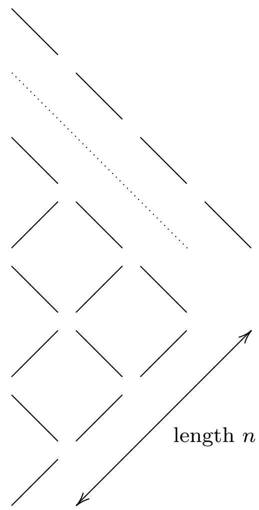
Let P be a graded poset with rank function $r : P \to N$. Let a_i be the number of order ideals of P of size i. The *rank-generating function* of $J(P)$ is $F(J(P), q) := \sum_{i \geq 0} a_i q^i$. We say P is *nice* if

$$\sum_{i \geq 0} a_i q^i = \prod_{p \in P} \frac{1 - q^{r(p)+2}}{1 - q^{r(p)+1}}$$

Let $\mathbf{n}$ represent the n-element chain for $n \in \mathbb{Z}_{>0}$. Prove that the following posets are nice:

(a) P=J($\mathbf{2}\times\mathbf{n}$) for $n \in \mathbb{Z}_{>0}$.

Proof. $J(\mathbf{2} \times \mathbf{n})$ has the structure



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Let $f(n) = \sum a_i q^i$ and $g(n) = \prod_{p \in J(\mathbf{2}\times\mathbf{n})} \dfrac{1 - q^{r(p)+2}}{1 - q^{r(p)+1}}$ We can easily verify that $f(1) = g(1)$, since the down sets of $J(\mathbf{2} \times \mathbf{1})$ are equivalent to the chain $\mathbf{4}$. Then $f(1) = 1 + q + q^2 + q^3$ and $g(1) = \left(\frac{1-q^2}{1-q}\right)\left(\frac{1-q^3}{1-q^2}\right)\left(\frac{1-q^4}{1-q^3}\right) = \frac{1-q^4}{1-q}$. Now, assume that $f(n) = g(n)$ and look at $f(n+1)$ and $g(n+1)$. To calculate $g(n+1)$, we add another layer to the diagram above. This creates extra vertices in our poset at ranks $n+1$ through $2n+2$. Then

$$g(n+1) = g(n)\left(\prod_{i=n+1}^{2n+2} \frac{1 - q^{i+2}}{1 - q^{i+1}}\right).$$

We see that the denominator of each term will cancel with the numerator of the preceding term, leaving only the last numerator and the first denominator, so this reduces to

$$\begin{aligned} g(n+1) &= g(n)\left(\frac{1 - q^{2n+2}}{1 - q^{n+1}}\right) \\ &= g(n)\left(1 + q^{n+1}\right). \end{aligned}$$

Next, we turn our attention to $f(n+1)$. We need to see which down sets have been added to $J(J(\mathbf{2} \times \mathbf{n{+}1}))$. Every down set that appears in $J(J(\mathbf{2} \times \mathbf{n}))$ will also appear in $J(J(\mathbf{2} \times \mathbf{n{+}1}))$, so we get $f(n)$ down sets that do not include the new layer. Now, if we assume that the right most vertex in the above picture is included in our down set, then we must also include all vertices along the bottom edge. The remaining information in the down sets that contain this point is contained in the triangular lattice above the bottom edge. But this also looks like $J(\mathbf{2} \times \mathbf{n{+}1})$, except that each down set has an extra $n+1$ vertices included. Then this contributes a term $f(n)q^{n+1}$. Since $f(n) = g(n)$, we get

$$\begin{aligned} f(n+1) &= f(n) + f(n)q^{n+1} \\ &= f(n)(1+q^{n+1}) \\ &= g(n)(1+q^{n+1}) \\ &= g(n+1) \end{aligned}$$

By induction, we get $f(n) = g(n)$ for all n.

(b) P=$\mathbf{m}\times\mathbf{n}$ for $n \in \mathbb{Z}_{>0}$.

Proof. We use a similar induction to part (a). Note that in this case, f and g are functions of two variables. Calculating $f(1,1)$ and $g(1,1)$, we get

$$\begin{aligned} f(1,1) &= 1+q, \\ g(1,1) &= \frac{1-q^2}{1-q} \\ &= 1+q. \end{aligned}$$

Now, assume that $f(h,k) = g(h,k)$ for all $h+k < K$. Then when $m+n = K$, we can calculate $f(m,n)$ and $g(m,n)$ recursively as follows:
Looking at the lattice $\mathbf{m}\times\mathbf{n}$, we can compute $g(m,n)$ by removing the last layer along the m direction to see a copy of **m-1** $\times$ **n** sitting inside $\mathbf{m}\times\mathbf{n}$. Then we multiply by $\frac{1-q^{r(p)+2}}{1-q^{r(p)+1}}$ for each vertex along the top edge, canceling denominators with the preceding numerators to get the expression

$$g(m,n) = g(m-1,n)\left(\frac{1-q^{m+n}}{1-q^m}\right).$$

We can derive a similar expression in terms of $g(m,n-1)$ by peeling off the last layer in the n direction to get

$$g(m,n) = g(m,n-1)\left(\frac{1-q^{m+n}}{1-q^n}\right).$$

Solving for $g(m,n-1$, we get

$$g(m,n-1) = g(m-1,n)\left(\frac{1-q^n}{1-q^m}\right).$$

Turning to $f(m,n)$, we again break into two cases. If the vertex in the $(m,1)$ position is not included in the down set, then no vertex in the m^{th} row is. These vertices contribute $f(m-1,n)$ to our sum. If the $(m,1)$ vertex is included in our down set, then the entire $(x,1)$ layer is included, so we get a contribution of $f(m,n-1)q^m$.

Then

$$\begin{aligned}
f(m,n) &= f(m-1,n) + f(m,n-1)q^m \\
&= g(m-1,n) + g(m,n-1)q^m \\
&= g(m-1,n)\left(1 + \frac{q^m(1-q^n)}{1-q^m}\right) \\
&= g(m-1,n)\left(\frac{1-q^m+q^m-q^{m+n}}{1-q^m}\right) \\
&= g(m-1,n)\left(\frac{1-q^{m+n}}{1-q^m}\right) \\
&= g(m,n).
\end{aligned}$$

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