Problem 4

(a) To show it is a Hopf algebra, we must show it is an algebra (associative, unital), that it is a coalgebra (coassociative, counital), that the algebra and coalgebra operations are compatible (via showing that Δ and ϵ are algebra homomorphisms), and that the antipode exists.

- Associativity. Let $x = x_1 \otimes \cdots \otimes x_m$, $y = y_1 \otimes \cdots \otimes y_n$, and $z = z_1 \otimes \cdots \otimes z_p$:

$$m(x, m(y,z)) = x_1 \otimes \cdots \otimes x_m \otimes y_1 \otimes \cdots \otimes y_n \otimes z_1 \otimes \cdots \otimes z_p = m(m(x,y), z)$$

- Unital:

$$u(1) \cdot x = 1 \cdot x = x = x \cdot 1 = x \cdot u(1)$$

- Coassociativity:

$$\begin{aligned}
(\Delta \otimes Id)\Delta x &= (\Delta \otimes Id) \sum_{S \subseteq [n]} x|_S \otimes x|_{[n] \setminus S} \\
&= \sum_{T \subseteq S} \sum_{S \subseteq [n]} x|_T \otimes x|_{S \setminus T} \otimes x|_{[n] \setminus S} \\
&= \sum_{A \sqcup B \sqcup C = [n]} x|_A \otimes x|_B \otimes x|_C \\
&= \sum_{T \subseteq S} \sum_{S \subseteq [n]} x|_{[n] \setminus S} \otimes x|_T \otimes x|_{S \setminus T} \\
&= (Id \otimes \Delta) \sum_{S \subseteq [n]} x|_{[n] \setminus S} \otimes x|_S \\
&= (Id \otimes \Delta)\Delta x
\end{aligned}$$

- Counital:

$$\begin{aligned}
(\epsilon * Id)(x) &= \sum_{S \subseteq [n]} \epsilon(x|_S)\, x|_{[n] \setminus S} \\
&= \epsilon(x|_\varnothing)\, x|_{[n]} \\
&= x
\end{aligned}$$

  and similarly $Id * \epsilon = Id$.
- Δ is an algebra homomorphism. Let $x = x_1 \otimes \cdots \otimes x_m$ and $y = y_1 \otimes \cdots \otimes y_n$.

$$\begin{aligned}
\Delta(xy) &= \sum_{S \subseteq [n+m]} (xy)|_S \otimes (xy)|_{[n+m] \setminus S} \\
&= \sum_{S \subseteq [n+m]} \left(x|_{S \cap [n]} \cdot y|_{S \cap [n]^C - n} \right) \otimes \left(x|_{[n] \cap S^C} \cdot y|_{([n+m] \setminus S) \cap [n]^C - n} \right) \\
&= \sum_{S \subseteq [m]} \sum_{T \subseteq [n]} (x|_S \cdot y|_T) \otimes \left(x|_{[m] \setminus S} \cdot y|_{[n] \setminus T} \right) \\
&= \sum_{S \subseteq [m]} \sum_{T \subseteq [n]} \left(x|_S \otimes x|_{[m] \setminus S} \right) \left(y|_T \otimes y|_{[n] \setminus T} \right) \\
&= \left(\sum_{S \subseteq [m]} \left(x|_S \otimes x|_{[m] \setminus S} \right) \right) \left(\sum_{T \subseteq [n]} \left(y|_T \otimes y|_{[n] \setminus T} \right) \right) \\
&= \Delta x \Delta y
\end{aligned}$$

- ϵ is an algebra homomorphism. Let x and y be as before.

$$\begin{aligned}
\epsilon(xy) &= \epsilon(x_1 \otimes \cdots \otimes x_m \otimes y_1 \otimes \cdots \otimes y_n) \\
&= \begin{cases} 1 & n = m = 0 \\ 0 & \text{otherwise} \end{cases} \\
&= \epsilon(x)\,\epsilon(y)
\end{aligned}$$

  So $T(V)$ is in fact a bialgebra.
- The antipode in this case is surprisingly simple. First, since $1 = u\epsilon(1) = (S * Id)1 = S(1) \cdot 1$, then $S(1) = 1$. Let $x = x_1$ be any pure 1-tensor, so that

$$0 = u\epsilon(x) = (S * Id)(x) = S(x)1 + S(1)x$$

  implying that

$$S(x) = -x.$$

Because S is an antimorphism, and because any m-tensor $x = x_1 \otimes \cdots \otimes x_m$ is the product $x_1 \ldots x_m$ of m 1-tensors,

$$\begin{aligned} S(x) &= S(x_1 \otimes \cdots \otimes x_m) \\ &= S(x_1 \cdots x_m) \\ &= S(x_m) \cdots S(x_1) \\ &= (-x_m) \cdots (-x_1) \\ &= (-1)^m (x_m \otimes \cdots \otimes x_1) \end{aligned}$$

(b) EXTERIOR ALGEBRA:
For $\bigwedge(V) = T(V)/I$ to be a Hopf algebra descended from $T(V)$, the ideal I must be a Hopf ideal, and this is only true if the characteristic of the base field is 2. Remember, I is the ideal generated by all $x \otimes x$ for $x \in V$.

To show that the ideal I is also a co-ideal, we must show that $\epsilon(I) = 0$ and $\Delta(I) \subset I \otimes T(V) + T(V) \otimes I$. The fact that $\epsilon(I) = 0$ is immediate from the fact that I is generated by 2-tensors, so it cannot contain any 0-tensors. For $L, R \in T(V)$ and $x \in V$ let

$$L \cdot x \otimes x \cdot R \in I$$

represent an arbitrary element of I. Then

$$\begin{aligned} \Delta(L \cdot x \otimes x \cdot R) &= \Delta L \Delta(x \otimes x) \Delta R \\ &= \Delta L \left((x \otimes x) \otimes 1 + \underbrace{2x \otimes x}_{=0} + 1 \otimes (x \otimes x) \right) \Delta R \\ &= \Delta L ((x \otimes x) \otimes 1 + 1 \otimes (x \otimes x)) \Delta R \\ &\in I \otimes T(V) + T(V) \otimes I. \end{aligned}$$

To show that $S(I) \subseteq I$, again consider arbitrary $L \otimes x \otimes x \otimes R \in I$. Then

$$\begin{aligned} S(L \cdot x \otimes x \cdot R) &= S(R) \cdot S(x \otimes x) \cdot S(L) \\ &= (-1)^2 S(R) \cdot x \otimes x \cdot S(L) \\ &\in I. \end{aligned}$$

So I is indeed a Hopf ideal and $\bigwedge(V)$ is a Hopf algebra.

SYMMETRIC ALGEBRA:
For $S(V) = T(V)/J$ is also a Hopf algebra where J is the ideal generated by $x \otimes y - y \otimes x$ for all $x, y \in V$. J is a coideal because

$$\epsilon(I) = 0$$

since J is generated by 2-tensors and so cannot contain any 0-tensors, and also because for any arbitrary element $L \cdot (x \otimes y - y \otimes x) \cdot R \in J$

$$\begin{aligned} \Delta(L \cdot (x \otimes y - y \otimes x) \cdot R) &= \Delta L \Delta(x \otimes y - y \otimes x) \Delta R \\ &= \Delta L ([(x \otimes y) \otimes 1 + x \otimes y + y \otimes x + 1 \otimes (x \otimes y)] - [(y \otimes x) \otimes 1 + y \otimes x + x \otimes y + 1 \otimes (y \otimes x)]) \Delta R \\ &= \Delta L [(x \otimes y - y \otimes x) \otimes 1] \Delta R + \Delta L [1 \otimes (x \otimes y - y \otimes x)] \Delta R \\ &\in J \otimes T(V) + T(V) \otimes J. \end{aligned}$$

It is a Hopf ideal because again for any $L \cdot (x \otimes y - y \otimes x) \cdot R \in J$,

$$\begin{aligned} S(L \cdot (x \otimes y - y \otimes x) \cdot R) &= S(R) \cdot S(x \otimes y - y \otimes x) \cdot S(L) \\ &= S(R) \cdot (Sy \otimes Sx - Sx \otimes Sy) \cdot S(L) \\ &= S(R) \cdot (y \otimes x - x \otimes y) \cdot S(L) \\ &\in J. \end{aligned}$$