3 Antipodes of incidence Hopf algebras have order 2

Notice that $S(S(P))$ is a sum of sequences of the form $[0, x_1] \times [x_1, x_2] \ldots \times [x_{n-1}, 1]$ with some sign. The coefficient of $P = [0, 1]$ is $(-1)^2 = 1$. We show that the coefficient of $[0, x_1] \times [x_1, x_2] \ldots \times [x_{n-1}, 1]$ is 0 for $n > 1$. In order to get that term, we must refine another subsequence $[0, y_1] \times [y_1, y_2] \ldots \times [y_{k-1}, 1]$. The sign of such refinement is $(-1)^{k+n}$. We say a subsequence is *nice* if $[0, y_1] = [0, x_1]$. We associate to it the subsequence $[0, y_2] \times [y_2, y_3] \ldots \times [y_{k-1}, 1]$, which has the opposite sign. It is easy to see that this is matching between nice and non-nice subsequences, so all terms cancel.