(2) (Computing antipodes)

(a) Compute the antipode of the *Hopf algebra of symmetric functions* described in Lectures 14 and 15.

Solution. (Worked with Karen and Seth) Let L_p be a chain with p edges. We use Takeuchi's formula to find the antipode:

$$S = \sum_{n \geq 0} (-1)^n m^{n-1} \pi^{\otimes n} \Delta^{n-1}(L_p)$$

Since $\Delta(L_p) = \sum_{i=0}^{p} L_i \otimes L_{p-1}$, applying Δ $n-1$ times gives:

$$\Delta^{n-1}(L_p) = \sum_{a_1+a_2+\cdots+a_n=p} L_{a_1} \otimes L_{a_2} \otimes \cdots \otimes L_{a_n}$$

Since π removes degree 0s, we restrict the above to be $0 < a_i \leq p$. Thus the antipode is

$$\begin{aligned} S &= \sum_{n\geq 0}(-1)^n \sum_{a_1+a_2+\cdots+a_n=p} L_{a_1} \times L_{a_2} \times \cdots \times L_{a_p} \\ &= \sum_{n\geq 0} \sum_{a_1+a_2+\cdots+a_n=p} (-1)^n L_{a_1} \times L_{a_2} \times \cdots \times L_{a_p} \end{aligned}$$

where $0 < a_i \leq p$. Notice further since $L_a \times L_b = L_b \times L_a$, we can "combine like terms", thus now we have

$$S = \sum_{n\geq 0} \sum_{a_1+a_2+\cdots+a_n=p} (-1)^n \binom{p}{a_1, a_2, \ldots, a_p} L_{a_1} \times L_{a_2} \times \cdots \times L_{a_p}$$

(b) Compute the antipode of the *Hopf algebra of non-commutative symmetric functions* described in Lecture 15.

Solution. (Worked with Karen and Seth) The difference from part a is that here $L_a \times L_b \neq L_b \times L_a$, so we cannot combine like terms. Thus

$$S = \sum_{n\geq 0} \sum_{a_1+a_2+\cdots+a_n=p} (-1)^n L_{a_1} \times L_{a_2} \times \cdots \times L_{a_p}$$