(a) Prove that the product of two Boolean posets is a Boolean poset.

Solution. Note that the Boolean poset is isomorphic to $2^S = \{\text{subsets of S}\}$. Let 2^S and 2^L be two distinct Boolean posets, then we want to show that $2^S \times 2^L \cong 2^{S \cup L}$ where S and L are disjoint. So we have that

$$2^S = \{S_1, S_2, \ldots, S_n \mid S_1 < S_2 < \ldots < S_n\},$$

which is all the sets of S ordered by containment. Similar we have,

$$2^L = \{L_1, L_2, \ldots, L_m \mid L_1 < L_2 < \ldots < L_m\},$$

the sets of L ordered by containment. Thus, when we take the product of two Boolean posets we obtain:

$$2^S \times 2^L = \{(S_1, L_1), (S_1, L_2), \ldots, (S_1, L_m), \ldots, (S_n, L_1), \ldots, (S_n, L_m)\}.$$

Now note that their are n-elements in 2^S and m elements in 2^L. Thus when we obtain the product we indeed obtain nm-elements.
Note that for

$$2^{S \cup L} = \{S_1 \cup L_1, S_1 \cup L_2, \ldots, S_n \cup L_1, S_n \cup L_2, \ldots, S_n \cup L_m\},$$

we obtain the direct products of unions, which also yields nm-elements. Let $\phi : 2^S \times 2^L \mapsto 2^{S \cup L}$ defined by $\phi(S_i, L_i) \mapsto S_i \cup L_i$. Now, we want to demonstrate that this is indeed a homomorphism.

$$\phi\left((S_i, L_j) + (S_k, L_l)\right) = \phi(S_i \cup S_k, L_j \cup L_l) = S_i \cup S_k \cup L_j \cup L_l$$

Additionally,

$$\phi(S_i, L_j) + \phi(S_k, L_l) = S_i \cup L_j \cup S_k \cup L_l = S_I \cup S_k \cup L_j \cup L_l$$

Now, we want to show that ϕ is injective. Assume $\phi(S_i, L_j) == \phi(S_k, L_l)$ then we have that $S_i \cup L_j = S_k \cup L_l$ which means that $S_i = S_k$ and $L_j = L_L$. This means that $i = k$ and $j = l$. This means that ϕ is injective. Now, noted that $2^S \times 2^L$ has the same cardinality as $2^{S \cup L}$. Thus, we see that ϕ is a bijective map. Therefore, we obtain that indeed $2^S \times 2^L \cong 2^{S \cup L}$.

(b) Let a *box* poset be a finite product of finite chains. Prove that any interval of a box poset is a box poset.

Solution. Let $B^{(n)}$ be the finite product of finite n chains. Note that we can write the elements of $B^{(n)}$ in terms of $a_1, a_2, \ldots, a_n)$. Where each a_i is an element of finite chain, C_i. So, now we can look at the interval $[(a_1, \ldots, a_n), (b_1, \ldots, b_n)]$ then $a \leq b_1, a_2 \leq b_2, \ldots, a_n \leq b_n$. So we see that from a_i to b_i we have a chain of $b_i - a_i$, which corresponds to C_i where $i = 1, \ldots, n$. Therefore, we see that $C_1 \times C_2 \times \ldots \times C_n$ is the product of chains from the interval $[(a_1, \ldots, a_n), (b_1, \ldots, b_n)]$.