PROBLEM 6

(a) I will show how to find the addition formula. First, we note that

$$\begin{aligned}
[f(x)]^n &= x^n + (\text{higher order terms}) \\
f^n(x) f^m(y) &= x^n y^m + (\text{higher order terms, } x^p x^q, p > n, q > m) \\
f(x+y) &= x + y + (\text{higher order terms}).
\end{aligned}$$

To get $f(x+y) = F(f(x), f(y))$, we need to match the coefficients

$$\left[x^i y^j\right] f(x+y) = \left[x^i y^j\right] F(f(x), f(y)),$$

for all i and j, both not equal to zero (here the expression on the right is expanded completely). This is is easy if we start with i and j small and work our way up. Let

$$\lambda_{i,j} = \left[f^i(x) f^j(y)\right] F(f(x), f(y))$$

so that the λ's are the coefficients of the non-expanded versions of powers of $f(x)$ and $f(y)$. Starting with $i = 1$ and $j = 0$ we see

$$\lambda_{1,0} = [x] F(f(x), f(y)) = [x] f(x+y) = 1$$

and vice-versa

$$\lambda_{0,1} = [y] F(f(x), f(y)) = [y] f(x+y) = 1$$

so far we know that

$$F(f(x), f(y)) = x + y + (\text{higher order terms}),$$

which is we can write as a function of $f(x)$ and $f(y)$ by observing that

$$\begin{aligned}
F(f(x), f(y)) &= x + y + (\text{higher order terms}) \\
&= f(x) + f(y) + (\text{same higher order terms}) - (\text{higher order terms of } f(x) + f(y)) \\
&= \lambda_{1,0} f(x) + \lambda_{0,1} f(y) + (\text{combined higher order terms}).
\end{aligned}$$

Since we are trying to create a series equal to $f(x+y)$, then

$$\begin{aligned} f(x+y) &= \lambda_{1,0} f(x) + \lambda_{0,1} f(y) + \text{(combined higher order terms)} \\ f(x+y) - \lambda_{1,0} f(x) - \lambda_{0,1} f(y) &= \text{(combined higher order terms)} \end{aligned}$$

We use this to proceed to values of i and j such that $i+j=2$. That is,

$$\begin{aligned} \lambda_{2,0} &= \left[x^2\right] F(f(x), f(y)) = \left[x^2\right] f(x+y) - f(x) - f(y) \\ \lambda_{1,1} &= [xy] F(f(x), f(y)) = [xy] f(x+y) - f(x) - f(y) \\ \lambda_{0,2} &= \left[y^2\right] F(f(x), f(y)) = \left[y^2\right] f(x+y) - f(x) - f(y) \end{aligned}$$

and actually, this is redundant because $f(y)$ doesn't have any x^2 terms, and the same goes for $f(x)$, more simply this is:

$$\begin{aligned} \lambda_{2,0} &= \left[x^2\right] F(f(x), f(y)) = \left[x^2\right] f(x+y) - f(x) \\ \lambda_{1,1} &= [xy] F(f(x), f(y)) = [xy] f(x+y) - f(x) - f(y) \\ \lambda_{0,2} &= \left[y^2\right] F(f(x), f(y)) = \left[y^2\right] f(x+y) - f(y) \end{aligned}$$

We can follow this procedure indefinitely and recursively for find all $\lambda_{i,j}$, noting that only the coefficients of lower order terms are needed (so $\lambda_{i,j}$ only depends on $\lambda_{m,n}$ where neither $m > i$, nor $n > j$). The recursive formula is:

$$\lambda_{i,j} = \left[x^i y^j\right] f(x+y) - \sum_{\substack{m \leq i \\ n \leq j}} \lambda_{n,m} f^n(x) f^m(y)$$

Thus the addition formula exists and is unique since all the choices we made were necessary. It is

$$F(f(x), f(y)) = \sum_{i,j \in \mathbb{N}} \lambda_{i,j} f^i(x) f^j(y)$$

or

$$F(x, y) = \sum_{i,j \in \mathbb{N}} \lambda_{i,j} x^i y^j.$$