5 a (With Federico Castillo)

We show

$$S(G) = \sum_{N \subseteq E} a(G/N)\, G|_N$$

where the sum is over some subsets N of edges of G, G/N is the graph obtained by contracting all edges in N (i.e identifying each edge with a single vertex), and $G|_N$ is the subgraph obtained by restricting the edges to N.

$|a(G)|$ counts the number of acyclic orientations of G.

$$S(G) = \sum_{n \geq 0} (-1)^n m^{n-1} \pi^{\otimes n} \Delta^{n-1}(G)$$

The map π removes all empty terms in the sum.

Let V be the set of vertices of G

We say $V \vDash (V_1, V_2, \ldots V_k)$ if $V = V_1 \sqcup V_2 \ldots \sqcup V_k$ and $V_i \neq \emptyset$

Then

$$S(G) = \sum_{V \vDash (V_1, \ldots, V_k)} (-1)^k (G|_{V_1} \sqcup G|_{V_2} \sqcup \ldots \sqcup G|_{V_k})$$

Clearly, $G|_{V_1} \sqcup \ldots \sqcup G|_{V_k}$ is a subgraph of G, associated with some set of edges N.

We need to count the coefficient for each N.

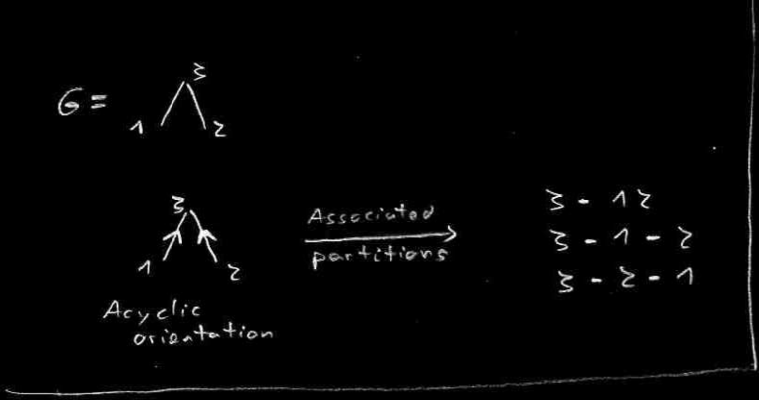
For each partition $V \vDash (V_1, \ldots, V_k)$ we associate the following orientation of the edges in G/N

If e is an edge in G/N, e joins two vertices $v_i \in V_i$ and $v_j \in V_j$ for some $i < j$. We orient e such that $v_i < v_j$.

Such an orientation is acyclic, as it is a partial order

Analogously, for any acyclic orientation one can find a partition $V \vDash (V_1, \ldots, V_k)$, by reconstructing the partial order in all relevant vertices.

We just need to prove that the coefficient for each acyclic orientation is 1 (but for a sign which depends on the size of N)



Let a be an acyclic orientation of G/N. The orientation imposes the order of some vertices in the partition.
N imposes that some vertices must lie in the same subset of the partition.
However, there are some pairs which may lie in any order in the partition, as there is no edge connecting them.

The picture shows the most simple example.
We notice the signs of the partition are such that the sum is 1, as we wanted.
By consecutely removing all such "indepent pairs" we conclude that each acyclic orientation is counted once in the summation.
This completes the proof.

5 6 (with Esteban Gonzalez)

It is straightforward to prove it is a Hopf algebra:

$$\Delta(\emptyset) = \emptyset \otimes \emptyset$$

$$\Delta(\pi_1 \pi_2) = \sum_{A \subseteq [n+m]} st(\pi_1\pi_2|_A) \otimes st(\pi_1\pi_2|_{[n+m]\setminus A})$$

$$= \sum_{\substack{A_1 \subseteq [n] \\ A_2 \subseteq [m]}} st(\pi_1|_{A_1}\, \pi_2|_{A_2}) \otimes st(\pi_1|_{[n]\setminus A_1}\, \pi_2|_{[m]\setminus A_2})$$

$$\epsilon(\emptyset) = 1$$

$$\epsilon(\pi_1 \pi_2) = \epsilon(\pi_1)\epsilon(\pi_2) = \begin{cases} 1, & \pi_1 = \pi_2 = \emptyset \\ 0, & \text{otherwise} \end{cases}$$

As it is graded with n, it is a Hopf algebra.

As in the previous exercise, the antipode is given by

$$S(\pi) = \sum_{[n] \vDash (I_1, \dots, I_k)} (-1)^k st(\pi|_{I_1}) \cdot st(\pi|_{I_2}) \cdots st(\pi|_{I_k})$$

We show there is a strong connection with the previous exercise.

For a permutation π, we associate a graph G_π with vertex set $[n]$, in which $i<j$ are adjacent if and only if $\pi(i) > \pi(j)$ (i.e. they are reversed)

For instance, $G_{(1,2,3,\dots,n)}$ is the graph on $[n]$ with no edges and $G_{(n,n-1,\dots,2,1)}$ is the complete graph K_n

Notice that $G_\pi|_I = G_{\pi|_I}$

We also have that $G_{\pi_1} \sqcup G_{\pi_2}$ is isomorphic to $G_{\pi_1 \pi_2}$

This is because any reversed pair in π_1 remains reversed in $\pi_1\pi_2$, any reversed pair in π_2 turns into a reversed pair in $\pi_1\pi_2$, but changing the labels.

Therefore, the Hopf algebra of permutations is "almost" the same as the Hopf algebra of graphs, but for some labels which are changed.

In particular, the antipode of permutations is similar to the antipode for graphs.

For instance, the coefficient of the identity permutation in $S(\pi)$ is the same coefficient of the graph with no edges in $S(G_\pi)$, i.e. the number of acyclic orientations of G_π

this is because labels are not important in the graph with no edges