$\square$

(b) Prove that the two-sided ideal $\langle xy - 1, yx - 1\rangle$ is a biideal of F, and therefore the quotient $H = F/\langle xy - 1, yx - 1\rangle$ is a bialgebra.

Proof. $J = \langle xy - 1, yx - 1\rangle$ is an ideal by construction. Then we must verify that J is a coideal. First, we check that $J \subseteq \ker \epsilon$. Given $h \in J$, we have $h = a(xy-1)+b(yx-1)+(xy-1)c+(yx-1)d$ for $a, b, c, d \in F$. Then

$$
\begin{aligned}
\epsilon(h) &= \epsilon(a(xy - 1) + b(yx - 1) + (xy - 1)c + (yx - 1)d) \\
&= \epsilon(a)(\epsilon(xy) - \epsilon(1)) + \epsilon(b)(\epsilon(yx) - \epsilon(1)) \\
&\quad + (\epsilon(xy) - \epsilon(1))\epsilon(c) + (\epsilon(yx) - \epsilon(1))\epsilon(d) \\
&= \epsilon(a)(1 - 1) + \epsilon(b)(1 - 1) + (1 - 1)\epsilon(c) + (1 - 1)\epsilon(d) \\
&= 0.
\end{aligned}
$$

Next, we verify that $\Delta(J) \subseteq J \otimes F + F \otimes J$. Observe that

$$\begin{aligned}
\Delta(xy-1) &= \Delta(x)\Delta(y) - \Delta(1) \\
&= (x \otimes x)(y \otimes y) - 1 \otimes 1 \\
&= xy \otimes xy - 1 \otimes 1 \\
&= \frac{1}{2}[xy \otimes xy + xy \otimes 1 - 1 \otimes xy - 1 \otimes 1 \\
&\qquad + xy \otimes xy - xy \otimes 1 + 1 \otimes xy - 1 \otimes 1] \\
&= \frac{1}{2}[(xy-1) \otimes (xy+1) + (xy+1) \otimes (xy-1)] \\
&\in J \otimes F + F \otimes J
\end{aligned}$$

This also works for $(yx-1)$,

$$\begin{aligned}
\Delta(yx-1) &= \Delta(x)\Delta(y) - \Delta(1) \\
&= \frac{1}{2}[(yx-1) \otimes (yx+1) + (yx+1) \otimes (yx-1)] \\
&\in J \otimes F + F \otimes J
\end{aligned}$$

Since J is an ideal, (as is F), then $h \otimes k(\Delta(xy-1)) \in J \otimes F + F \otimes J$ for any $f, k \in F$, as is $(\Delta(xy-1))h \otimes k$, and similarly for $\Delta(yx-1)$. Then, apply Δ to an element of J.

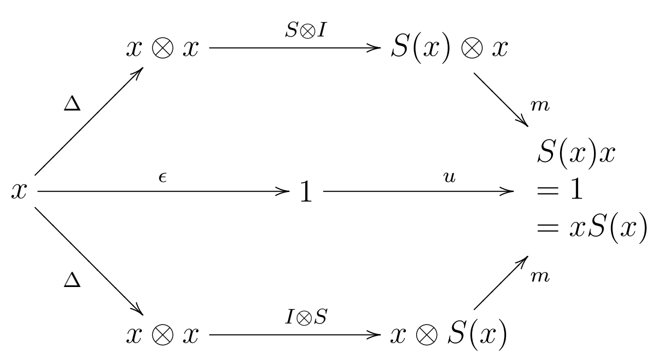
$$\begin{aligned}
&\Delta(a(xy-1)b + c(yx-1)d) \\
&\quad = \Delta(a)(\Delta(xy-1))\Delta(b) + \Delta(c)(\Delta(yx-1))\Delta(d) \\
&\quad \in F \otimes J + J \otimes F.
\end{aligned}$$

This verifies that J is a coideal, so J is a biideal. Thus $H = F/J$ is a bialgebra.

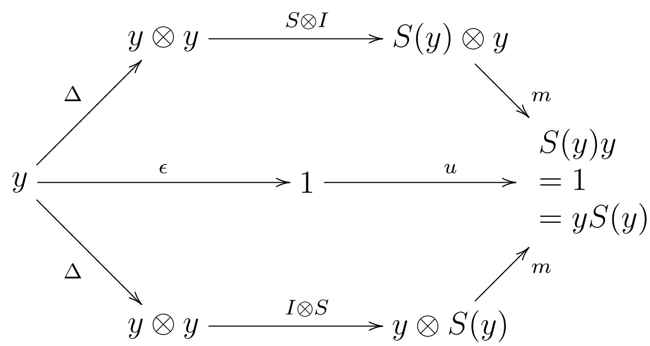
□

(c) Prove that H is a Hopf algebra by finding its (unique) antipode S. Find the order of S.

Proof.

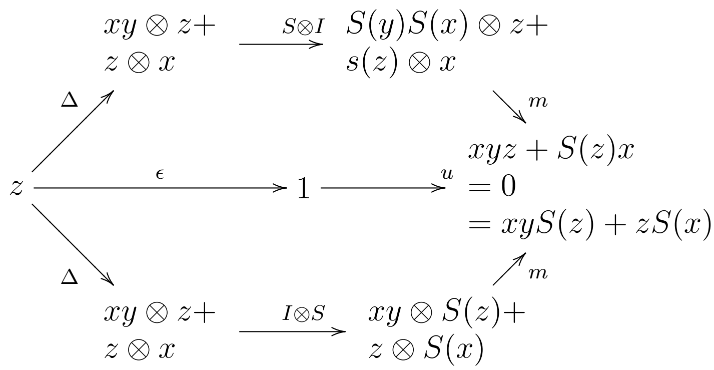


Thus, we see that $S(x) = y$.



S(y)=x

To calculate $S(z)$, we use the relation $xy = 1$ in the coproduct of z.



Since $xy = 1$, from the bottom equality $S(z) + zS(x) = 0$, we get $S(z) = -zy$. Plugging into the top equality, we get $z - zyx = z - z = 0$. Then we have a well defined antipode given by

$$S(x) = y$$
$$S(y) = x$$
$$S(z) = -zy.$$

We see that $S^2(x) = S(y) = x$, $S^2(y) = S(x) = y$. What remains is to compute the order of S with respect to z:

$$\begin{aligned} S(z) &= -zy \\ S^2(z) &= xzy \\ S^3(z) &= -xzy^2 \\ S^4(z) &= x^2zy^2 \\ &\vdots \\ S^n(z) &= (-1)^n x^{\lfloor \frac{n}{2} \rfloor} z y^{\lceil \frac{n}{2} \rceil}. \end{aligned}$$

The computations here have been omitted for the sake of brevity. The key to understanding this formula is to note that the antipode

is an antihomomorphism. That is, for $a, b \in H$, $S(ab) = S(b)S(a)$. Given this, and the fact that $S(z) = -zy$ and S turns xs into ys and vice versa, we see that for a word of the form $x^m zy^n$, $S(x^m zy^n) = (S(y))^n S(z)(S(x))^m = -x^n zy^{m+1}$. Then with each application of S, all xs on the left become ys on the right, and vice versa, and we add one extra y on the right side of z.
Since we are in a noncommutative algebra, and our only relations are $xy = yx = 1$, we see that $(-1)^n x^{\lfloor \frac{n}{2} \rfloor} zy^{\lceil \frac{n}{2} \rceil} = z$ if and only if $n = 0$. Then the order of S in H is infinite.

□

(d) Prove that the two-sided ideal $\langle x^n - 1 \rangle$ is a Hopf ideal of H, and therefore $J = H/\langle x^n - 1 \rangle$ is a Hopf algebra.

Proof. We check that $K = \langle x^n - 1 \rangle$ is a biideal: Let $h \in K$. Then $h = a(x^n - 1) + (x^n - 1)b$ for some $a, b \in H$.

$$\begin{aligned} \epsilon(h) &= \epsilon(a(x^n - 1) + (x^n - 1)b) \\ &= \epsilon(a)(\epsilon(x)^n - 1) + (\epsilon(x)^n - 1)\epsilon(b) \\ &= \epsilon(a)(0) + (0)\epsilon(b) \\ &= 0, \end{aligned}$$

so $K \subseteq \ker \epsilon$.

$$\begin{aligned} \Delta(h) &= \Delta(a(x^n - 1)b) \\ &= \Delta(a)(\Delta(x^n) - \Delta(1))\Delta(b) \\ &= \Delta(a)(x^n \otimes x^n - 1 \otimes 1)\Delta(b) \\ &= \Delta(a)\left(\frac{1}{2}\left[(x^n - 1) \otimes (x^n + 1) + (x^n + 1) \otimes (x^n - 1)\right]\right)\Delta(b) \\ &\in H \otimes K + K \otimes H. \end{aligned}$$

To show that the biideal is also a Hopf ideal, we need to show that $S(K) \subseteq K$.

$$\begin{aligned} S(h) &= S(a(x^n - 1) + (x^n - 1)b) \\ &= S(a)(S(x^n) - S(1)) + (S(x^n) - S(1))S(b) \\ &= S(a)(y^n - 1) + (y^n - 1)S(b) \\ &= S(a)(x^n - 1)(-y^n) + (x^n - 1)(-y^n)S(b) \\ &\in K. \end{aligned}$$

This completes the verification that K is a Hopf ideal, so H/K is a Hopf algebra.

□

(e) Prove that the antipode of J has order $2n$.

Proof. Note that our calculations from part (c) still hold after modding out by K, so we still have

$$
\begin{aligned}
S(x) &= y \\
S(y) &= x \\
S(z) &= -zy.
\end{aligned}
$$

We still have $S^2(x) = x$ and $S^2(y) = y$, so we need to recalculate the order of S on z. Our relations are $xy = yx = x^n = 1$. Also, we can use the identity $yx = 1$ and $x^n = 1$ to get $1 = y^n$.

$$
\begin{aligned}
S^k(z) &= (-1)^k x^{\lfloor \frac{k}{2} \rfloor} z y^{\lceil \frac{k}{2} \rceil} \\
&\overset{set}{=} z.
\end{aligned}
$$

Then

$$
\begin{aligned}
(-1)^k &= 1, \\
x^{\lfloor \frac{k}{2} \rfloor} &= 1, \\
y^{\lceil \frac{k}{2} \rceil} &= 1.
\end{aligned}
$$

By $(-1)^k = 1$, we know $k = 2m$ for some $m \in \mathbb{Z}$, so $\lfloor \frac{k}{2} \rfloor = \lceil \frac{k}{2} \rceil = m$. Then we are looking for the least m such that $x^m = y^m = 1$. This gives $m = n$, so the order of S on z is $2n$. Since $S^2(x) = x$ and $S^2(y) = y$, then $S^{2n}(x) = x$ and $S^{2n}(y) = y$, so the order of S is $2n$.

□