3. Prove the following identities for f, g, h in a Hopf Algebra H.

ⓐ $\sum_{(h)} h_{(1)} S(h_{(2)}) \otimes h_{(3)} = h$

$\sum_{(h)} h_{(1)} S(h_{(2)}) \otimes h_{(3)} = \sum_{(h)} \varepsilon(h) \otimes h_{(3)} = \sum_{(h)} 1_H \otimes h_{(3)} = 1_H \otimes \sum_{(h)} h_{(1)} = 1_H \otimes h$

Since $H \xrightarrow{\cong} \{1_H\} \otimes H$ by $h \mapsto 1_H \otimes h$ for $h \in H$

we have that $\sum h_{(1)} S(h_{(2)}) \otimes h_{(3)} = h$

ⓑ $\sum_{(g)(h)} h_{(1)} S(g_{(1)} f h_{(2)}) g_{(2)} = \varepsilon(gh) S(f)$

$\sum_{(g)(h)} h_{(1)} S(g_{(1)} f h_{(2)}) g_{(2)} = \sum_{(g)(h)} h_{(1)} S(h_{(2)}) S(f) S(g_{(1)}) g_{(2)}$

$= \sum_{(g)} \varepsilon(h) S(f) S(g_{(1)}) g_{(2)} = \left(\sum_{(g)} S(g_{(1)}) g_{(2)}\right) \varepsilon(h) S(f) = \varepsilon(g)\varepsilon(h) S(f) = \varepsilon(gh) S(f)$

ⓒ $\sum_{(h)} (1 \otimes S(h_{(3)}) h_{(1)}) \Delta S(h_{(2)}) = (S \otimes S)\Delta(h)$

$\sum_{(h)} (1 \otimes S(h_{(3)}) h_{(1)}) \Delta S(h_{(2)}) = \sum_{(h)} (1 \otimes S(h_{(3)}) h_{(1)}) \left(\sum_{(h_{(2)})} S(h_{(2)}) \otimes S(h_{(1)})\right)$

$= \sum_{(h_{(2)})(h)} S(h_{(2)}) \otimes S(h_{(3)}) h_{(1)} S(h_{(1)}) = \sum_{(h_{(2)})(h)} S(h_{(2)}) \otimes S(h_{(3)}) \varepsilon(h) = \sum_{(h_{(2)})(h)} S(h_{(2)}) \otimes S(h_{(3)})$

and

$(S \otimes S)\Delta(h) = \sum_{(h)} S(h_{(1)}) \otimes S(h_{(2)}) = \sum_{(h_2)(h)} S(h_{(2)}) \otimes S(h_{(3)})$