Problem 2

Proposition. *The finite dimensional coalgebra* $(\mathbb{H}^*, \Delta, \epsilon)$ *with dual basis* $\{1^*, i^*, j^*, k^*\}$ *has the co-product defined by:*

$$\begin{aligned} \Delta 1^* &= 1^* \otimes 1^* - i^* \otimes i^* - j^* \otimes j^* - k^* \otimes k^* \\ \Delta i^* &= 1^* \otimes i^* + i^* \otimes 1^* + j^* \otimes k^* - k^* \otimes j^* \\ \Delta j^* &= 1^* \otimes j^* + j^* \otimes 1^* + k^* \otimes i^* - i^* \otimes k^* \\ \Delta k^* &= 1^* \otimes k^* + k^* \otimes 1^* + i^* \otimes j^* - j^* \otimes i^*. \end{aligned}$$

Proof. (Worked with Maria and Brian) Let $e_0 = 1$, $e_1 = i$, $e_2 = j$, and $e_3 = k$, and since $\mathbb{H}$ is finite dimensional, the dual $\mathbb{H}^*$ has a basis of functionals $\{e_0^*, e_1^*, e_2^*, e_3^*\}$ defined on the basis of $\mathbb{H}$ by

$$e_k^*(e_j) = \delta_{jk} = \begin{cases} 1 & j = k \\ 0 & j \neq k \end{cases}.$$

Let $\rho : \mathbb{H}^* \otimes \mathbb{H}^* \to (\mathbb{H} \otimes \mathbb{H})^*$ be the injective linear map we defined earlier as $\langle \rho(g^* \otimes h^*), a \otimes b\rangle = \langle g^*, a\rangle \langle h^*, b\rangle$. The coproduct $\Delta : \mathbb{H}^* \to \mathbb{H}^* \otimes \mathbb{H}^*$ then must be defined using the dual of the multiplication of the algebra m^* with the formula

$$\Delta = \rho^{-1} m^*.$$

Since

$$\Delta e_k^* = \sum_{i,j \le 3} \lambda_{ij}^{(k)} e_i^* \otimes e_j^*$$

for some scalars $\lambda_{ij}^{(k)}$, then to find the values of these scalars we observe that by applying ρ to both sides,

$$\begin{aligned} \rho \Delta e_k^* &= \rho \sum_{i,j \le 3} \lambda_{ij}^{(k)} e_i^* \otimes e_j^* \\ \rho \rho^{-1} m^* e_k^* &= \sum_{i,j \le 3} \lambda_{ij}^{(k)} \rho\left(e_i^* \otimes e_j^*\right) \\ m^* e_k^* &= \sum_{i,j \le 3} \lambda_{ij}^{(k)} \rho\left(e_i^* \otimes e_j^*\right). \end{aligned}$$

We evaluate these functionals on the basis of $\mathbb{H} \otimes \mathbb{H}$, so for all $p, q \le 3$

$$\begin{aligned} \langle m^* e_k^*, e_p \otimes e_q\rangle &= \left\langle \sum_{i,j \le 3} \lambda_{ij}^{(k)} \rho\left(e_i^* \otimes e_j^*\right), e_p \otimes e_q \right\rangle \\ \langle e_k^*, m(e_p \otimes e_q)\rangle &= \sum_{i,j \le 3} \lambda_{ij}^{(k)} \langle e_i^*, e_p\rangle \langle e_j^*, e_q\rangle \\ \langle e_k^*, m(e_p \otimes e_q)\rangle &= \lambda_{pq}^{(k)} \end{aligned}$$

where $\langle e_k^*, m(e_p \otimes e_q)\rangle$ is the e_k-coordinate of $e_p \otimes e_q$. This means that

$$\Delta e_k^* = \sum_{i,j \le 3} (\text{the } e_k\text{-coordinate of } e_i e_j)\, e_i^* \otimes e_j^*.$$

For example, $\Delta 1^*$ only includes tensors of the form $e_i^* \otimes e_j^*$ where $e_i e_j$ equals 1 or -1. This is only true for $1 \cdot 1 = 1$, $i \cdot i = -1$, $j \cdot j = -1$, and $k \cdot k = -1$, so

$$\Delta 1^* = 1^* \otimes 1^* - i^* \otimes i^* - j^* \otimes j^* - k^* \otimes k^*.$$

The rest are shown similarly. □

Proposition. 1^* *is a cocommutative element that generates a subcoalgebra that is not commutative.*

Proof. Since $\Delta 1^* = 1^* \otimes 1^* - i^* \otimes i^* - j^* \otimes j^* - k^* \otimes k^*$, the element $1^* \in \mathbb{H}^*$ satisfies $T\Delta 1^* = \Delta 1^*$ and is thus cocommutative. Since $\Delta 1^*$ introduces the terms i^*, j^*, and k^*, it generates the whole coalgebra. So since it contains the element i^*, which is noncocommutative because

$$\Delta i^* = 1^* \otimes i^* + i^* \otimes 1^* + j^* \otimes k^* - k^* \otimes j^*$$

the generated (trivial) subcoalgebra is noncocommutative. □

Problem 3

(a) (Worked with Brian and Maria) Using Sweedler notation we have

$$\begin{aligned} \sum_{(h)} \left(h_{(1)} S\left(h_{(2)}\right) \otimes h_{(3)}\right) &= \sum_{(h)} \left(u\epsilon\left(h_{(1)}\right) \otimes h_{(2)}\right) \\ &= \sum_{(h)} \left(\epsilon\left(h_{(1)}\right) \otimes h_{(2)}\right) \\ &= h. \end{aligned}$$

where the first equality is given by a modified form of the antipode diagram