Problem 1. Let P be a finite poset and $A(P)$ be its incidence algebra. Recall that the identity is the function $\mathbf{1} \in A(P)$ defined by $\mathbf{1}([x,x]) = 1$ for all x in P and $\mathbf{1}([x,y]) = 0$ for all $x < y$ in P. Define the *zeta function* of P by $\zeta([x,y]) = 1$ for all $x \leq y$ in P.

(a) Prove that $f \in A(P)$ has a two-sided inverse if and only if $f([x,y]) \neq 0$ for all $x \in P$.

(b) By part (a), $(2-\zeta)$ is invertible in $A(P)$. Prove that

$$(2-\zeta)^{-1}([x,y]) = (\text{\# of chains in } P \text{ from } x \text{ to } y).$$

(c) By part (a), ζ is invertible in $A(P)$. Its inverse is $\mu = \zeta^{-1}$ is call the *Mobius function* of P. Prove the *Mobius inversion formula*:
Let $f : P \to V$ and $g : P \to V$ be functions for P to a vector space V. Then

$$g(y) = \sum_{x \leq y} f(x) \quad \text{for all } y \in P$$

if and only if

$$f(y) = \sum_{x \leq y} \mu(x,y) g(x) \quad \text{for all } y \in P$$

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Solution. *Crista Moreno, Karla Lanzas, Nina Cerutti, Stephen Collazos*

(a) Suppose f has a two-sided inverse g. Then $f * g = \mathbf{1} = g * f$ so that

$$f * g([x,x]) = f([x,x])g([x,x]) = \mathbf{1}([x,x]) = 1.$$

Therefore $f([x,x]) \neq 0$.
Conversely suppose $f([x,x]) \neq 0$. I will show that we can define a function g such that $f * g([x,y]) = \mathbf{1}([x,y])$ for any $x \leq y \in P$ by induction on the length of the maximal chain in $[x,y]$. For any $x \in P$, let $g([x,x]) = \dfrac{1}{f([x,x])}$. Then

$$f * g([x,x]) = f([x,x])g([x,x]) = f([x,x])\frac{1}{f([x,x])} = 1 = \mathbf{1}([x,x]).$$

Similarly $g*f([x,x])=1$. Now for an interval $[x,y]$ where y covers x, let $g([x,y])=\dfrac{-f([x,y])}{f([x,x])f([y,y])}$. We can define g in this way because $f([x,x])f([y,y])\neq 0$. Then

$$\begin{aligned}
f*g([x,y]) &= f([x,x])g([x,y])+f([x,y])g([y,y])\\
&= f([x,x])\frac{-f([x,y])}{f([x,x])f([y,y])}+f([x,y])\frac{1}{f(y,y)}=0\\
&= \mathbf{1}([x,y])
\end{aligned}$$

and similarly $g*f([x,y])=0$. Suppose we can define g for any interval with maximal chain of length n. Consider an interval $[x,y]$ with a maximal chain of length $n+1$ Then let $g([x,y])=\dfrac{\sum_{x<z\leq y}f([x,z])g([z,y])}{f([x,x])}$. Then

$$\begin{aligned}
f*g([x,y]) &= \sum_{x\leq z\leq y} f([x,z])g([z,y])\\
&= f([x,x])g([x,y])+\sum_{x<z\leq y} f([x,z])g([z,y])=0\\
&= \mathbf{1}([x,y])
\end{aligned}$$

and similarly $g*f([x,y])=0$. Therefore, g is a two-sided inverse of f.

(b) First consider the function $(\zeta-1)$:

$$(\zeta-1)([x,y])=\begin{cases}1 & x<y\\ 0 & x=y\end{cases}$$

For positive integers k we see that,

$$\begin{aligned}
(\zeta-1)^k([x,y]) &= \sum_{x=x_0<x_1<\cdots<x_k=y}\prod_{i=0}^{k-1}(\zeta-1)([x_i,x_{i+1}])\\
&= \sum_{x=x_0<x_1<\cdots<x_k=y} 1\\
&= \text{the number of chains of length } k \text{ in } [x,y]
\end{aligned}$$

Now let l be the length of the maximal chain of $[x,y]$. Then certainly, $(\zeta-1)^{l+1}([u,v])=0$ for all $x\leq u\leq v\leq y$ because $[x,y]$ has 0 chains of length $l+1$. Therefore we can observe

$$\begin{aligned}
&(2-\zeta)\left(1+(\zeta-1)+(\zeta-1)^2+\cdots+(\zeta-1)^l\right)([x,y])\\
&=\left(1-(\zeta-1)\right)\left(1+(\zeta-1)+(\zeta-1)^2+\cdots+(\zeta-1)^l\right)([x,y])\\
&=\left(1-(\zeta-1)^{l+1}\right)([x,y])=\mathbf{1}([x,y])
\end{aligned}$$

This shows that $(2-\zeta)^{-1}=1+(\zeta-1)+(\zeta-1)^2+\cdots(\zeta-1)^l$. In other words, $(2-\zeta)^{-1}$ is the number of chains of length 0+ the number of chains of length 1+ the number of chains of length $2+\cdots+$ the number of chains of lenth $l=$ the total number of chains in P from x to y.

(c) Suppose $g(y) = \sum_{x \le y} f(x)$ for all $y \in P$. Then

$$\sum_{x \le y} \mu([x,y]) g(x) = \sum_{x \le y} \mu([x,y]) \left(\sum_{z \le x} f(z) \right) = \sum_{z \le x \le y} \mu([x,y]) f(z)$$

$$= \sum_{z \le x \le y} \mu([x,y]) \zeta([z,x]) f(z) \quad \text{because } \zeta([z,x]) = 1 \text{ for all } z \le x \in P$$

$$= \sum_{z \le y} f(z) \left(\sum_{z \le x \le y} \zeta([z,x]) \mu([x,y]) \right)$$

$$= \sum_{z \le y} f(z) \underbrace{\mathbf{1}([z,y])}_{= \begin{cases} 1 & z = y \\ 0 & z < y \end{cases}}$$

$$= f(y)$$

Conversely, suppose $f(y) = \sum_{x \le y} \mu(x,y) g(x)$. Then

$$\sum_{x \le y} f(x) = \sum_{x \le y} \left(\sum_{z \le x} \mu([z,x]) g(z) \right)$$

$$= \sum_{z \le x \le y} \mu([z,x]) \zeta([x,y]) g(z)$$

$$= \sum_{z \le y} \underbrace{\mathbf{1}([z,y])}_{= \begin{cases} 1 & z = y \\ 0 & z < y \end{cases}} g(z)$$

$$= g(y)$$