6 Incidence coalgebras distinguish posets

First, we show that $x \in I(P)$ satisfies $\Delta(x) = x \otimes x$ if and only if $x = [a, a]$ for some $a \in P$. Let $x = \sum r_i[a_i, b_i]$ with $r_i \in \mathbb{F}, a_i, b_i \in P$. Then

$$\Delta(x) = \sum r_i \Delta[a_i b_i] = \sum r_i \sum_{a_i \leq z_i \leq b_i} [a_i, z_i] \otimes [z_i, b_i] \tag{1}$$

$$x \otimes x = \sum r_i r_j [a_i b_i] \otimes [a_j b_j] \tag{2}$$

Notice that if $a_i < b_i$, then $[a_i, b_i] \otimes [a_i, b_i]$ appears in equation 2 but not in 1, so $a_i = b_i$ for all i. We easily conclude that $x = [a, a]$ in the same way as in Problem 3 of Homework 1. Therefore we can identify all the poset elements in $I(P)$.

We say that an element b *covers* a if $a < b$ and there is no c such that $a < c < b$. We now prove that b covers a if and only if there exists $x \in I(P)$ such that

$$\Delta(x) = [a, a] \otimes x + x \otimes [b, b]$$

If b covers a, then $x = [a, b]$. If x exists, assume $x = \sum r_i[a_i, b_i]$, then

$$[a, a] \otimes x + x \otimes [b, b] = \sum r_i[a, a] \otimes [a_i, b_i] + r_i[a_i, b_i] \otimes [b, b] \tag{3}$$

Comparing equations 1 and 3 we conclude $a = a_i$ and $b_i = b$, so $x = [a, b]$. Finally, if b did not cover a there would be more terms in $\Delta(x)$, which is a contradiction.

As we can identify the poset elements and we can recover all covering relations, then the whole poset structure is uniquely determined by $I(P)$.