Problem 4

Lemma. *If X is a subspace of V and*

$$\sum_{i=1}^{n} v_i \otimes w_i \in X \otimes X$$

where $v_i \in V$ are non-zero and $w_i \in V$ are linearly independent, then all the w_i's are elements of X. The same is true if we switch the left and right sides of the tensor. Therefore, if $\{v_i\}$ and $\{w_i\}$ are both linearly independent, then both sets are contained in X.

Proof. Let $\{x_i\}_{i\in I}$ be a basis for X so that there exist $\beta_{ij} \in \mathbb{F}$ such that

$$\sum_{k=1}^{n} v_k \otimes w_k \quad = \quad \sum_{i,j\in I} \beta_{ij} x_i \otimes x_j \in X \otimes X.$$

If all the w_k's are in X, we are done; so instead suppose that some nonempty subset of them are outside of X. We can then move then tensors from the left-hand side of the equation for which $w_k \in X$ to the right side, and have them be absorbed into the other sum. That is, if

$$w_k = \sum_{j\in I} \lambda_j^{(k)} x_j, \text{ for } w_k \in X$$

then

$$\begin{aligned}
\sum_{k,\, w_k \notin X} v_k \otimes w_k &= \sum_{i,j\in I} \beta_{ij} x_i \otimes x_j - \sum_{k,\, w_k\in X} v_k \otimes w_k \\
&= \sum_{i,j\in I} \beta_{ij} x_i \otimes x_j - \sum_{k,\, w_k\in X} v_k \otimes \sum_{j\in I} \lambda_j^{(k)} x_j \\
&= \sum_{j\in I} \left(\sum_{i\in I} \beta_{ij} x_i \right) \otimes x_j - \sum_{j\in I} \left(\sum_{k,\, w_k\in X} \lambda_j^{(k)} v_k \right) \otimes x_j \\
&= \sum_{j\in I} \left(\sum_{i\in I} \beta_{ij} x_i - \sum_{k,\, w_k\in X} \lambda_j^{(k)} v_k \right) \otimes x_j.
\end{aligned}$$

We can subtract the beginning from the end to get

$$0 = \sum_{j\in I} \left(\sum_{i\in I} \beta_{ij} x_i - \sum_{k,\, w_k\in X} \lambda_j^{(k)} v_k \right) \otimes x_j - \sum_{k,\, w_k\notin X} v_k \otimes w_k$$

but since the set $\{w_k : w_k \notin X\} \cup \{x_j\}_{j\in I}$ is linearly independent, all the left-hand sides of the tensors must be zero, a contradiction since none of the v_k's are zero. Thus, all of the w_k's are inside of X. The proof is identical for the case in which the linearly independent elements are on the left instead of the right. And of course if they're on both sides the proof can be applied twice to show that all of the elements are in X. □

Proposition. *All the non-trivial subpsaces $\{\mathbb{F}V : V \subseteq S, V \neq \emptyset\}$ are subcoalgebras of $\langle \mathbb{F}S, \Delta, \epsilon\rangle$ with the inherited operations.*

Proof. First, they are all co-algebras as they are of the form $\mathbb{F}V$ where V is a set with $\Delta_{\mathbb{F}V}(v) = v \otimes v$ and $\epsilon_{\mathbb{F}V}(v) = 1$, for all $v \in V$, and these operations are inherited as restrictions of Δ and ϵ to $\mathbb{F}V$.

To show they are the only subcoalgebras contained in $\mathbb{F}S$, it is only necessary to show that for any subalgebra $X \subseteq \mathbb{F}S$, there exists $S' \subseteq S$ such that $\mathbb{F}S' = X$.

Let $\{x_i\}_{i\in I}$ be a basis for X so that

$$x_i = \sum_{s\in S'} \lambda_s^{(i)} s$$

is the expansion of x_i in terms of the elements of $S' \subseteq S$, the smallet subset of S needed to write out the x_i's so that for some $s \in S'$ not all $\lambda_s^{(i)} = 0$. So far we know that $X \subseteq \mathbb{F}S'$.

Let's show inclusion in the other direction. The coproduct $\Delta_X : X \to X \otimes X$ is inherited from the coproduct $\Delta : \mathbb{F}S \to \mathbb{F}S \otimes \mathbb{F}S$ so that $\Delta_X(x) = \Delta(x)$; thus,

$$\begin{aligned}
\Delta_X(x_i) &= \Delta(x_i) \\
&= \Delta \sum_{s\in S} \lambda_s^{(i)} s \\
&= \sum_{s\in S'} \lambda_s^{(i)} s \otimes s \in X \otimes X.
\end{aligned}$$

By the preceeding lemma, this means all of the $s \in S'$ for which $\lambda_s^{(i)} \neq 0$ are in X. Since for all $s \in S'$, there exists some $j \in I$ such that $\lambda_s^{(i)} \neq 0$, then all $s \in S'$ are in X. So $\mathbb{F}S' \subseteq X$.

Ergo, $\mathbb{F}S' = X$. $\square$