(2) (a) Let $T : C \otimes C \to C \otimes C$ be the *twist* map given by $T(c \otimes d) = d \otimes c$ for $c, d \in C$. Show that T is a bilinear map and write $T\Delta(c)$ in Sweedler notation.

Proof. We begin by defining $\tilde{T}$ a bilinear map from $C \times C \to C \otimes C$ by defining $\tilde{T}$ on the basis elements of $C \times C$ and extending linearly. For c_i, c_j basis elements of C, define $\tilde{T}(c_i, c_j) = c_j \otimes c_i$. Then moding out by the relations of $C \otimes C$, $\tilde{T}$ induces a linear map $T : C \otimes C \to C \otimes C$ by $T(c \otimes d) = d \otimes c$ for all pure tensors $c \otimes d \in C \otimes C$, and extending linearly over sums. In Sweedler notation, we write

$$\begin{aligned} T\Delta(c) &= T \sum_{(c)} c_{(1)} \otimes c_{(2)} \\ &= \sum_{(c)} T(c_{(1)} ox c_{(2)}) \\ &= \sum_{(c)} c_{(2)} \otimes c_{(1)} \end{aligned}$$

□

(b) Prove the following identities:

(i) $\Delta(c) = \sum_{(c)} \epsilon(c_{(2)}) \otimes \Delta(c_{(1)})$

Proof. We begin by proving a lemma.

Lemma 1. *For linear functions $f : C \to C$ and $g : C \to C$, $T \circ (f \otimes g) = (g \otimes f) \circ T$.*

Proof of lemma: Let $\sum_i x_i \otimes y_i \in C \otimes C$. Then by linearity of T f and g,

$$\begin{aligned} T \circ (f \otimes g)\left(\sum_i x_i \otimes y_i\right) &= T \circ \left(\sum_i f(x_i) \otimes g(y_i)\right) \\ &= \sum_i T(f(x_i) \otimes g(y_i)) \\ &= \sum_i (g(y_i) \otimes f(x_i)) \\ &= \sum_i (g \otimes f)(y_i \otimes x_i) \\ &= \sum_i (g \otimes f) \circ T(x_i \otimes y_i) \\ &= (f \otimes g) \circ T\left(\sum_i x_i \otimes y_i\right). \end{aligned}$$

□

$$\begin{aligned} &\sum_{(c)} \epsilon(c_{(2)}) \otimes \Delta(c_{(1)}) \\ &= \sum_{(c)} (\epsilon \otimes \Delta)(c_{(2)} \otimes c_{(1)}) \\ &= (\epsilon \otimes \Delta)T(\Delta(c)) \\ &= (I \otimes \Delta)(\epsilon \otimes I)T(\Delta(c)) \\ &= (I \otimes \Delta)T(I \otimes \epsilon)(\Delta(c)) && \textit{by Lemma 1,} \\ &= (I \otimes \Delta)T(c \otimes 1) && \textit{by the counitary property,} \\ &= (I \otimes \Delta)(1 \otimes c) \\ &= 1 \otimes \Delta(c) \\ &\cong \Delta(c). \end{aligned}$$

□

(ii) $\Delta(c) = \sum_{(c)} c_{(1)} \otimes \epsilon(c_{(3)}) \otimes c_{(2)}$

Proof.

$$
\begin{aligned}
&\sum_{(c)} c_{(1)} \otimes \epsilon(c_{(3)}) \otimes c_{(2)} \\
&= (I \otimes \epsilon \otimes I) \sum_{(c)} c_{(1)} \otimes c_{(3)} \otimes c_{(2)} \\
&= (I \otimes \epsilon \otimes I)(I \otimes T)\Delta_2(c) \\
&= (I \otimes T)(I \otimes I \otimes \epsilon)(I \otimes \Delta)\Delta(c) \quad \textit{by Lemma } 1, \\
&= (I \otimes T)(I \otimes ((I \otimes \epsilon) \circ \Delta))\Delta(c) \\
&= (I \otimes T) \sum_{(c)} c_{(1)} \otimes c_{(2)} \otimes 1 \quad \textit{by the counitary property}, \\
&= \sum_{(c)} c_{(1)} \otimes 1 \otimes c_{(2)} \\
&\cong \Delta(c).
\end{aligned}
$$

□

(iii) $c = \sum_{(c)} \epsilon(c_{(1)}) \otimes \epsilon(c_{(3)}) \otimes c_{(2)}$

Proof.

$$
\begin{aligned}
&\sum_{(c)} \epsilon(c_{(1)}) \otimes \epsilon(c_{(3)}) \otimes c_{(2)} \\
&= (\epsilon \otimes \epsilon \otimes I) \sum_{(c)} c_{(1)} \otimes c_{(3)} \otimes c_{(2)} \\
&= (\epsilon \otimes \epsilon \otimes I)(I \otimes T) \sum_{(c)} c_{(1)} \otimes c_{(2)} \otimes c_{(3)} \\
&= (\epsilon \otimes \epsilon \otimes I)(I \otimes T)\Delta_2(c) \\
&= (I \otimes T)(\epsilon \otimes I \otimes \epsilon)(I \otimes \Delta)\Delta(c) \quad \textit{by Lemma } 1, \\
&= (I \otimes T)(\epsilon \otimes ((I \otimes \epsilon) \circ \Delta)\Delta(c) \\
&= (I \otimes T) \sum_{(c)} \epsilon(c_{(1)}) \otimes c_{(2)} \otimes 1 \quad \textit{by the counitary property}, \\
&= (I \otimes T)(\epsilon \otimes I \otimes I)(\Delta(c) \otimes 1) \\
&= (I \otimes T)((\epsilon \otimes I) \circ \Delta \otimes I)(c \otimes 1) \\
&= (I \otimes T)(1 \otimes c \otimes 1) \\
&= 1 \otimes 1 \otimes c \\
&\cong c.
\end{aligned}
$$