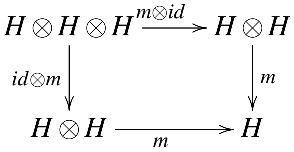
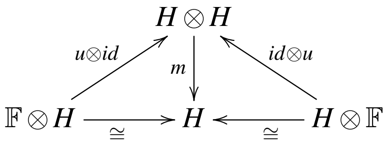
Solution. *Crista Moreno, Karla Lanzas*

$H = H_4$ is an $\mathbb{F}$-algebra so it has multiplication $m : H \otimes H \to H$ and unit $u : \mathbb{F} \to H$ such that the following diagrams commute:





Consider the coproduct $\Delta : H \to H \otimes H$ given by $\Delta(g) = g \otimes g$ and $\Delta(x) = x \otimes 1 + g \otimes x$ and the counit $\varepsilon : H \to \mathbb{F}$ given by $\varepsilon(g) = 1$ and $\varepsilon(x) = 0$. We can extend Δ so that $\Delta(1) = 1$ and $\Delta(gx) = gx \otimes g + 1 \otimes gx$. We see that Δ is multiplicative relative to the basis relations:

$$\Delta(g^2) = \Delta(g)\Delta(g) = (g \otimes g)(g \otimes g) = g^2 \otimes g^2 = 1 \otimes 1 = \Delta(1)$$

$$\begin{aligned}\Delta(x^2) &= \Delta(x)\Delta(x) = (x \otimes 1 + g \otimes x)(x \otimes 1 + g \otimes x) \\ &= x^2 \otimes 1 + xg \otimes x + gx \otimes x + g^2 \otimes x^2 \\ &= -(gx \otimes x) + gx \otimes x = 0 = \Delta(0)\end{aligned}$$

$$
\begin{aligned}
\Delta(gx) &= \Delta(g)\Delta(x) = (g\otimes g)(x\otimes 1 + g\otimes x)\\
&= gx\otimes g + 1\otimes gx = -xg\otimes g + 1\otimes -xg\\
&= -(xg\otimes g) - (1\otimes xg)\\
&= -(x\otimes 1 + g\otimes x)(g\otimes g) = -\Delta(x)\Delta(g) = \Delta(-xg)
\end{aligned}
$$

Similarly, we extend ε so that $\varepsilon(1) = 1$ and $\varepsilon(gx) = 0$ and we see that ε is multiplicative relative to the basis relations:

$$\varepsilon(g^2) = \varepsilon(g)\varepsilon(g) = 1\cdot 1 = 1 = \varepsilon(1)$$

$$\varepsilon(x^2) = \varepsilon(x)\varepsilon(x) = 0\cdot 0 = 0 = \varepsilon(x)$$

$$\varepsilon(gx) = \varepsilon(g)\varepsilon(x) = 1\cdot 0 = 0 = -0\cdot 1 = -\varepsilon(x)\varepsilon(g) = \varepsilon(-xg)$$

Now that we have Δ and ε maps defined on the basis of H that preserve the given relations, in order to make H a *coalgebra*, we need the following diagrams to commute:

We can observe how each diagram acts on each element of the basis $\{1, g, x, gx\}$ of H:
1:

$$
\begin{aligned}
(id\otimes\Delta)\Delta(1) &= (id\otimes\Delta)(1\otimes 1) = 1\otimes(1\otimes 1)\\
&= (1\otimes 1)\otimes 1 = \Delta(1)\otimes 1\\
&= (\Delta\otimes id)\Delta(1)
\end{aligned}
$$

$$
\begin{aligned}
(\varepsilon\otimes id)\Delta(1) &= (\varepsilon\otimes id)(1\otimes 1) = \varepsilon(1)\otimes 1 \cong 1\\
(id\otimes\varepsilon)\Delta(1) &= (id\otimes\varepsilon)(1\otimes 1) = 1\otimes\varepsilon(1) = 1\otimes 1 \cong 1
\end{aligned}
$$

g :

$$
\begin{aligned}
(id\otimes\Delta)\Delta(g) &= (id\otimes\Delta)(g\otimes g) = g\otimes(g\otimes g)\\
&= (g\otimes g)\otimes g = \Delta(g)\otimes g\\
&= (\Delta\otimes id)\Delta(g)
\end{aligned}
$$

$$
\begin{aligned}
(\varepsilon\otimes id)\Delta(g) &= (\varepsilon\otimes id)(g\otimes g) = \varepsilon(g)\otimes g = 1\otimes g \cong g\\
(id\otimes\varepsilon)\Delta(g) &= (id\otimes\varepsilon)(g\otimes g) = g\otimes\varepsilon(g) = g\otimes 1 \cong g
\end{aligned}
$$

x:

$$
\begin{aligned}
(id\otimes\Delta)\Delta(x) &= (id\otimes\Delta)(x\otimes 1+g\otimes x) = x\otimes\Delta(1)+g\otimes\Delta(x)\\
&= x\otimes(1\otimes 1)+g\otimes(x\otimes 1+g\otimes x) = x\otimes 1\otimes 1+g\otimes x\otimes 1+g\otimes g\otimes x\\
&= (x\otimes 1+g\otimes x)\otimes 1+(g\otimes g)\otimes x = \Delta(x)\otimes 1+\Delta(g)\otimes x\\
&= (\Delta\otimes id)(x\otimes 1+g\otimes x)\\
&= (\Delta\otimes id)\Delta(x)
\end{aligned}
$$

$$
\begin{aligned}
(\varepsilon\otimes id)\Delta(x) &= (\varepsilon\otimes id)(x\otimes 1+g\otimes x) = \varepsilon(x)\otimes 1+\varepsilon(g)\otimes x\\
&= 0\otimes 1+1\otimes x = 1\otimes x\cong x\\
(id\otimes\varepsilon)\Delta(x) &= (id\otimes\varepsilon)(x\otimes 1+g\otimes x) = x\otimes\varepsilon(1)+g\otimes\varepsilon(x)\\
&= x\otimes 1+g\otimes 0 = x\otimes 1\cong x
\end{aligned}
$$

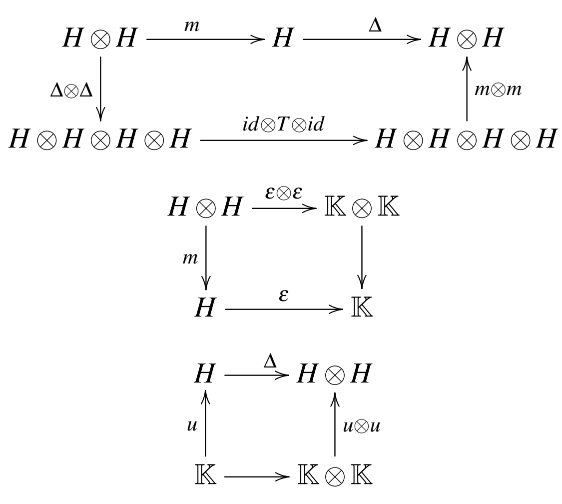
gx:

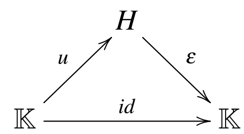
$$
\begin{aligned}
(id\otimes\Delta)\Delta(gx) &= (id\otimes\Delta)(gx\otimes g+1\otimes gx) = gx\otimes\Delta(g)+1\otimes\Delta(gx)\\
&= gx\otimes(g\otimes g)+1\otimes(gx\otimes g+1\otimes gx)\\
&= gx\otimes g\otimes g+1\otimes gx\otimes g+1\otimes 1\otimes gx\\
&= (gx\otimes g+1\otimes gx)\otimes g+(1\otimes 1)\otimes gx\\
&= \Delta(gx)\otimes g+\Delta(1)\otimes gx = (\Delta\otimes id)(gx\otimes g+1\otimes gx)\\
&= (\Delta\otimes id)\Delta(gx)
\end{aligned}
$$

$$
\begin{aligned}
(\varepsilon\otimes id)\Delta(gx) &= (\varepsilon\otimes id)(gx\otimes g+1\otimes gx) = \varepsilon(gx)\otimes g+\varepsilon(1)\otimes gx\\
&= 0\otimes g+1\otimes gx = 1\otimes gx\cong gx\\
(id\otimes\varepsilon)\Delta(gx) &= (id\otimes\varepsilon)(gx\otimes g+1\otimes gx) = gx\otimes\varepsilon(g)+x\otimes\varepsilon(gx)\\
&= gx\otimes 1+x\otimes 0 = gx\otimes 1\cong gx
\end{aligned}
$$

Hence the coassociative and conunitary maps commute for the basis elements. Therefore (H,Δ,ε) is a coalgebra.

In order for H to be a *bialgebra*, we need Δ and ε to be *algebra* maps. That is we must show the following four diagrams commute:





We can observe that the first map commutes the 16 basis elements of $H \otimes H$ with the following computations:

$$\begin{aligned}(m\otimes m)(id\otimes T\otimes id)(\Delta\otimes\Delta)(1\otimes 1) &= (m\otimes m)(id\otimes T\otimes id)(1\otimes 1\otimes 1\otimes 1)\\ &= (m\otimes m)(1\otimes 1\otimes 1\otimes 1) = 1\otimes 1 = \Delta(1)\\ &= \Delta m(1\otimes 1)\end{aligned}$$

$$\begin{aligned}(m\otimes m)(id\otimes T\otimes id)(\Delta\otimes\Delta)(1\otimes g) &= (m\otimes m)(id\otimes T\otimes id)(1\otimes 1\otimes g\otimes g)\\ &= (m\otimes m)(1\otimes g\otimes 1\otimes g) = g\otimes g = \Delta(g)\\ &= \Delta m(1\otimes g)\end{aligned}$$

$$\begin{aligned}(m\otimes m)(id\otimes T\otimes id)(\Delta\otimes\Delta)(1\otimes x) &= (m\otimes m)(id\otimes T\otimes id)((1\otimes 1)\otimes(x\otimes 1+g\otimes x))\\ &= (m\otimes m)(id\otimes T\otimes id)((1\otimes 1\otimes x\otimes 1)+(1\otimes 1\otimes g\otimes x))\\ &= (m\otimes m)(1\otimes x\otimes 1\otimes 1+1\otimes g\otimes 1\otimes x) = x\otimes 1+g\otimes x = \Delta(x)\\ &= \Delta m(1\otimes x)\end{aligned}$$

$$\begin{aligned}(m\otimes m)(id\otimes T\otimes id)(\Delta\otimes\Delta)(1\otimes gx) &= (m\otimes m)(id\otimes T\otimes id)((1\otimes 1)\otimes(gx\otimes g+1\otimes gx))\\ &= (m\otimes m)(id\otimes T\otimes id)((1\otimes 1\otimes gx\otimes g)+(1\otimes 1\otimes 1\otimes gx))\\ &= (m\otimes m)(1\otimes gx\otimes 1\otimes g+1\otimes 1\otimes 1\otimes gx) = gx\otimes g+1\otimes gx = \Delta(gx)\\ &= \Delta m(1\otimes gx)\end{aligned}$$

$$\begin{aligned}(m\otimes m)(id\otimes T\otimes id)(\Delta\otimes\Delta)(g\otimes 1) &= (m\otimes m)(id\otimes T\otimes id)(g\otimes g\otimes 1\otimes 1)\\ &= (m\otimes m)(g\otimes 1\otimes g\otimes 1) = g\otimes g = \Delta(g)\\ &= \Delta m(g\otimes 1)\end{aligned}$$

$$\begin{aligned}(m\otimes m)(id\otimes T\otimes id)(\Delta\otimes\Delta)(g\otimes g) &= (m\otimes m)(id\otimes T\otimes id)(g\otimes g\otimes g\otimes g)\\ &= (m\otimes m)(g\otimes g\otimes g\otimes g) = 1\otimes 1 = \Delta(1)\\ &= \Delta m(g\otimes g)\end{aligned}$$

$$\begin{aligned}(m\otimes m)(id\otimes T\otimes id)(\Delta\otimes\Delta)(g\otimes x) &= (m\otimes m)(id\otimes T\otimes id)((g\otimes g)\otimes(x\otimes 1+g\otimes x))\\ &= (m\otimes m)(id\otimes T\otimes id)(g\otimes g\otimes x\otimes 1+g\otimes g\otimes g\otimes x)\\ &= (m\otimes m)(g\otimes x\otimes g\otimes 1+g\otimes g\otimes g\otimes x) = gx\otimes g+1\otimes gx = \Delta(gx)\\ &= \Delta m(g\otimes x)\end{aligned}$$

$$\begin{aligned}(m\otimes m)(id\otimes T\otimes id)(\Delta\otimes\Delta)(g\otimes gx) &= (m\otimes m)(id\otimes T\otimes id)((g\otimes g)\otimes(gx\otimes g+1\otimes gx))\\ &= (m\otimes m)(id\otimes T\otimes id)(g\otimes g\otimes gx\otimes g+g\otimes g\otimes 1\otimes gx)\\ &= (m\otimes m)(g\otimes gx\otimes g\otimes g+g\otimes 1\otimes g\otimes gx) = x\otimes 1+g\otimes x = \Delta(x)\\ &= \Delta m(g\otimes gx)\end{aligned}$$

$$
\begin{aligned}
(m\otimes m)(id\otimes T\otimes id)(\Delta\otimes\Delta)(x\otimes 1) &= (m\otimes m)(id\otimes T\otimes id)((x\otimes 1+g\otimes x)\otimes 1\otimes 1)\\
&= (m\otimes m)(id\otimes T\otimes id)(x\otimes 1\otimes 1\otimes 1+g\otimes x\otimes 1\otimes 1\\
&= (m\otimes m)(x\otimes 1\otimes 1\otimes 1+g\otimes 1\otimes x\otimes 1) = x\otimes 1+g\otimes x = \Delta(x)\\
&= \Delta m(x\otimes 1)
\end{aligned}
$$

$$
\begin{aligned}
(m\otimes m)(id\otimes T\otimes id)(\Delta\otimes\Delta)(x\otimes g) &= (m\otimes m)(id\otimes T\otimes id)((x\otimes 1+g\otimes x)\otimes g\otimes g)\\
&= (m\otimes m)(id\otimes T\otimes id)(x\otimes 1\otimes g\otimes g+g\otimes x\otimes g\otimes g\\
&= (m\otimes m)(x\otimes g\otimes 1\otimes g+g\otimes g\otimes x\otimes g) = xg\otimes g+1\otimes xg = \Delta(xg)\\
&= \Delta m(x\otimes g)
\end{aligned}
$$

$$
\begin{aligned}
(m\otimes m)(id\otimes T\otimes id)(\Delta\otimes\Delta)(x\otimes x) &= (m\otimes m)(id\otimes T\otimes id)((x\otimes 1+g\otimes x)\otimes(x\otimes 1+g\otimes x)\\
&= (m\otimes m)(id\otimes T\otimes id)(x\otimes 1\otimes x\otimes 1+x\otimes 1\otimes g\otimes x+g\otimes x\otimes x\otimes 1+g\otimes x\otimes g\otimes x)\\
&= (m\otimes m)(x\otimes x\otimes 1\otimes 1+x\otimes g\otimes 1\otimes x+g\otimes x\otimes x\otimes 1+g\otimes g\otimes x\otimes x)\\
&= 0\otimes 1+xg\otimes x+gx\otimes x+1\otimes 0 = -gx\otimes x+gx\otimes x = \Delta(0)\\
&= \Delta m(x\otimes x)
\end{aligned}
$$

$$
\begin{aligned}
(m\otimes m)(id\otimes T\otimes id)(\Delta\otimes\Delta)(x\otimes gx) &= (m\otimes m)(id\otimes T\otimes id)((x\otimes 1+g\otimes x)\otimes(gx\otimes g+1\otimes gx)\\
&= (m\otimes m)(id\otimes T\otimes id)(x\otimes 1\otimes gx\otimes g+x\otimes 1\otimes 1\otimes gx+g\otimes x\otimes gx\otimes g\\
&\quad +g\otimes x\otimes 1\otimes gx)\\
&= (m\otimes m)(x\otimes gx\otimes 1\otimes g+x\otimes 1\otimes 1\otimes gx+g\otimes gx\otimes x\otimes g+g\otimes 1\otimes x\otimes gx)\\
&= 0\otimes g+x\otimes gx+x\otimes xg+g\otimes 0 = x\otimes gx-x\otimes gx = \Delta(0)\\
&= \Delta m(x\otimes gx)
\end{aligned}
$$

$$
\begin{aligned}
(m\otimes m)(id\otimes T\otimes id)(\Delta\otimes\Delta)(gx\otimes 1) &= (m\otimes m)(id\otimes T\otimes id)((gx\otimes g+1\otimes gx)\otimes 1\otimes 1)\\
&= (m\otimes m)(id\otimes T\otimes id)(gx\otimes g\otimes 1\otimes 1+1\otimes gx\otimes 1\otimes 1\\
&= (m\otimes m)(gx\otimes 1\otimes g\otimes 1+1\otimes 1\otimes gx\otimes 1) = gx\otimes g+1\otimes gx = \Delta(gx)\\
&= \Delta m(gx\otimes 1)
\end{aligned}
$$

$$
\begin{aligned}
(m\otimes m)(id\otimes T\otimes id)(\Delta\otimes\Delta)(gx\otimes g) &= (m\otimes m)(id\otimes T\otimes id)((gx\otimes g+1\otimes gx)\otimes g\otimes g)\\
&= (m\otimes m)(id\otimes T\otimes id)(gx\otimes g\otimes g\otimes g+1\otimes gx\otimes g\otimes g\\
&= (m\otimes m)(gx\otimes g\otimes g\otimes g+1\otimes g\otimes gx\otimes g) = gxg\otimes g+g\otimes gxg = \Delta(gxg)\\
&= \Delta m(gx\otimes g)
\end{aligned}
$$

$$
\begin{aligned}
(m\otimes m)(id\otimes T\otimes id)(\Delta\otimes\Delta)(gx\otimes x) &= (m\otimes m)(id\otimes T\otimes id)((gx\otimes g+1\otimes gx)\otimes x\otimes 1+g\otimes x)\\
&= (m\otimes m)(id\otimes T\otimes id)(gx\otimes g\otimes x\otimes 1+gx\otimes g\otimes g\otimes x+1\otimes gx\otimes x\otimes 1\\
&\quad +1\otimes gx\otimes g\otimes x\\
&= (m\otimes m)(gx\otimes x\otimes g\otimes 1+gx\otimes g\otimes g\otimes 1+1\otimes x\otimes gx\otimes 1+1\otimes g\otimes gx\otimes x)\\
&= 0\otimes g+-x\otimes g+x\otimes gx+g\otimes 0 = \Delta(0)\\
&= \Delta m(gx\otimes g)
\end{aligned}
$$

$$
\begin{aligned}
(m\otimes m)(id\otimes T\otimes id)(\Delta\otimes\Delta)(gx\otimes gx) &= (m\otimes m)(id\otimes T\otimes id)((gx\otimes g+1\otimes gx)\otimes gx\otimes g+1\otimes gx)\\
&= (m\otimes m)(id\otimes T\otimes id)(gx\otimes g\otimes gx\otimes g+gx\otimes g\otimes 1\otimes gx+1\otimes gx\otimes gx\otimes g\\
&\quad +1\otimes gx\otimes 1\otimes gx\\
&= (m\otimes m)(gx\otimes gx\otimes g\otimes g+gx\otimes 1\otimes g\otimes gx+1\otimes gx\otimes gx\otimes g+1\otimes 1\otimes gx\otimes gx)\\
&= 0\otimes 1+gx\otimes x+gx\otimes -x+1\otimes 0=\Delta(0)\\
&= \Delta m(gx\otimes gx)
\end{aligned}
$$

Hence, m **and** Δ **are compatible**. Now we observe the second map commutes with the 16 basis elements of $H\otimes H$ with the following computations:

$$(\varepsilon\otimes\varepsilon)(1\otimes 1)=(\varepsilon(1)\otimes\varepsilon(1))=1\otimes 1\cong 1=\varepsilon(1)=\varepsilon\circ m(1\otimes 1)$$

$$(\varepsilon\otimes\varepsilon)(1\otimes g)=(\varepsilon(1)\otimes\varepsilon(g))=1\otimes 1\cong 1=\varepsilon(g)=\varepsilon\circ m(1\otimes g)$$

$$(\varepsilon\otimes\varepsilon)(1\otimes x)=(\varepsilon(1)\otimes\varepsilon(x))=1\otimes 0\cong 0=\varepsilon(x)=\varepsilon\circ m(1\otimes x)$$

$$(\varepsilon\otimes\varepsilon)(1\otimes gx)=(\varepsilon(1)\otimes\varepsilon(gx))=1\otimes 0\cong 0=\varepsilon(gx)=\varepsilon\circ m(1\otimes gx)$$

$$(\varepsilon\otimes\varepsilon)(g\otimes 1)=(\varepsilon(g)\otimes\varepsilon(1))=1\otimes 1\cong 1=\varepsilon(g)=\varepsilon\circ m(g\otimes 1)$$

$$(\varepsilon\otimes\varepsilon)(g\otimes g)=(\varepsilon(g)\otimes\varepsilon(g))=1\otimes 1\cong 1=\varepsilon(g^2)=\varepsilon\circ m(g\otimes g)$$

$$(\varepsilon\otimes\varepsilon)(g\otimes x)=(\varepsilon(g)\otimes\varepsilon(x))=1\otimes 0\cong 0=\varepsilon(gx)=\varepsilon\circ m(g\otimes x)$$

$$(\varepsilon\otimes\varepsilon)(g\otimes gx)=(\varepsilon(g)\otimes\varepsilon(gx))=1\otimes 0\cong 0=\varepsilon(x)=\varepsilon\circ m(g\otimes gx)$$

$$(\varepsilon\otimes\varepsilon)(x\otimes 1)=(\varepsilon(x)\otimes\varepsilon(1))=0\otimes 1\cong 0=\varepsilon(x)=\varepsilon\circ m(x\otimes 1)$$

$$(\varepsilon\otimes\varepsilon)(x\otimes g)=(\varepsilon(x)\otimes\varepsilon(g))=0\otimes 1\cong 0=\varepsilon(xg)=\varepsilon\circ m(x\otimes g)$$

$$(\varepsilon\otimes\varepsilon)(x\otimes x)=(\varepsilon(x)\otimes\varepsilon(x))=0\otimes 0\cong 0=\varepsilon(x^2)=\varepsilon\circ m(x\otimes x)$$

$$(\varepsilon\otimes\varepsilon)(x\otimes xg)=(\varepsilon(x)\otimes\varepsilon(xg))=0\otimes 0\cong 0=\varepsilon(xxg)=\varepsilon\circ m(x\otimes xg)$$

$$(\varepsilon\otimes\varepsilon)(gx\otimes 1)=(\varepsilon(gx)\otimes\varepsilon(1))=0\otimes 1\cong 0=\varepsilon(gx)=\varepsilon\circ m(gx\otimes 1)$$

$$(\varepsilon\otimes\varepsilon)(gx\otimes g)=(\varepsilon(gx)\otimes\varepsilon(g))=0\otimes 1\cong 0=\varepsilon(gxg)=\varepsilon\circ m(gx\otimes 1)$$

$$(\varepsilon\otimes\varepsilon)(gx\otimes x)=(\varepsilon(gx)\otimes\varepsilon(x))=0\otimes 0\cong 0=\varepsilon(gx^2)=\varepsilon\circ m(gx\otimes x)$$

$$(\varepsilon \otimes \varepsilon)(gx \otimes gx) = (\varepsilon(gx) \otimes \varepsilon(gx)) = 0 \otimes 0 \cong 0 = \varepsilon((gx)^2) = \varepsilon \circ m(gx \otimes gx)$$

Hence, ε **and** m **are compatible**. Now we observe the third map commutes for any element $\lambda \in \mathbb{K}$:

$$\begin{aligned} \Delta \circ u(\lambda) &= \lambda(\Delta \circ u(1_{\mathbb{K}})) = \lambda(\Delta(1)) = \lambda(1 \otimes 1) \\ &= (u \otimes u)(\lambda(1_{\mathbb{K}} \otimes 1_{\mathbb{K}})) = (u \otimes u)(\lambda \otimes 1_{\mathbb{K}}) \\ &\text{note: } \lambda \otimes 1_{\mathbb{K}} \cong \lambda \end{aligned}$$

Hence, u **and** Δ **are compatible**. Finally, we observe the fourth map commutes for any element $\lambda \in \mathbb{K}$:

$$\begin{aligned} \varepsilon(u(\lambda)) &= \lambda(\varepsilon(u(1_{\mathbb{K}}))) = \lambda(\varepsilon(1)) = \lambda(1_{\mathbb{K}}) \\ &= \lambda \end{aligned}$$

Hence, ε **and** u **are compabtible**. We have that all four diagmrams commute so $(H, m, u, \Delta, \varepsilon)$ is a bialgebra.