7 A "Catalan algebra"[1]

A *n-lucky word* of *length* k is a sequence of consecutive numbers $(a, a+1, \ldots, a+k-1)$ in $\{1, 2, \ldots, n-1\}$. A *n-lucky sentence* is a sequence of n-lucky words $w_1, \ldots w_m$ such that for $1 \le l < m$

- the first term in w_l is strictly greater than the first term in w_{l+1}.
- the last term in w_l is strictly greater than the last term in w_{l+1}.

The proof has two main steps.

Step 1 We show $\dim A_n$ is the number of n-lucky sentences.

Step 2 We find a bijection between n-lucky sentences and dick paths on the $(n-1) \times (n-1)$ square. The cardinality of Dyck paths is known to be the catalan number.

Step 1

Clearly, A_n is generated by all possible finite products $\prod e_i$. We refer to the product $e_{i_1} e_{i_2} \ldots e_{i_k}$ simply as $i_1, i_2, \ldots, i_k$. We show we can simplify each of those products to a n-lucky sentence. We assume wlog $q = 1$.

Assume we have the number k in two different positions in the product. We refer to them as a *k-pair*. Consider a *minimal* k-pair, i.e. one which does not contain any other k'-pair between them. Consider the following cases.

- If there is no any $k-1, k+1$ in the middle of the k-pair, all terms inside commute with k, so we can move the left k to the right and cancel one k. We say we *killed* the k-pair.

[1] Joint work with Federico Castillo

- If there is a $k-1$ in the middle, but there is no any $k+1$, we can move both k's close to the $k-1$ and replace them with a single k. Again, we killed the k-pair. Notice there may not be more $k-1$'s in the middle as the pair is minimal. The same happens if there is a $k+1$ but no $k-1$.

- If there is a $k-1$ and a $k+1$, we say the k-pair is *strong*.

Let a be the greatest number in the product. We show we can kill all a-pairs. Consider an a-pair with no a's inside. As a is the greatest number, the a-pair is not strong. We can kill it, unless there is a $a-1$-pair inside it. Consider an $a-1$-pair with no other $a-1$ inside. There are no a's inside it, so we we can kill it unless there is a $a-2$-pair inside. As there are finitely possible terms, we can kill all such pairs.

Therefore, we may assume there is a single a in the product. We move it to the left until it is in the first position or until there it is next to an $a-1$. In the latter case, if we restrict to the left of a, then $a-1$ is the greatest number so we may assume it is the only one. We now move the lucky word $(a-1, a)$ to the left until it is in the first position or until it is next to an $a-2$. Again, we may assume it is the only one. We move the lucky word $(a-2, a-1, a)$ to the left and we go on.

After following these steps, our product begins with a lucky word $w_1 = (b_1, b_1+1, \dots, a_1)$. If we repeat this procedure to the terms on the right of a, we will get another lucky word $w_2 = (b_2, b_2+1, \dots, a_2)$ to the right of w_1, which satisfies $a_1 > a_2$. If $b_1 \leq b_2$, then $b_2 \in w_1$ as well and the b_2-pair can be killed. Therefore, we may assume $b_1 > b_2$. We go on and the product reduces to a lucky sentence.

Therefore, n-lucky sentences generate A_n. A lucky sentence cannot be further reduced as all pairs are strong, so they are a basis for A_n.

Step 2

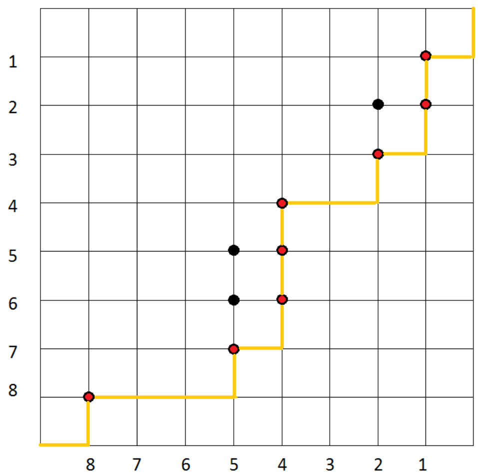

Figure 1: Dyck path corresponding to the lucky sentence $8 - 567 - 456 - 23 - 12$. Discarded points are shown black.

We recall a n-Dyck path is a staircase walk from $(0,0)$ to $(n-1,n-1)$ that lies strictly below (but may touch) the diagonal.

Let $(w_k,\ldots,w_2,w_1)$ be a n-lucky sentence with k words. Let a_i be the first and the first element of the i-th word, respectively. We now consider a rectangular grid $n\times n$ and we number the inner horizontal and vertical from 1 to $n-1$, as shown in Figure 1. For each word w_i, we mark a dot in the positions (a_i,x) for all $x\in w_i$. A dot is discarded if it is to the right of another dot. Finally, we construct a Dyck path which passes through all the remaining dots. The path goes up only if there is a dot in that line.

The path constructed is indeed a Dyck path, as there are no two points in any horizontal line, and all dots are below the diagonal by construction. It is easy to construct the inverse. We recover the discarded points by joining all corners to the diagonal, and it is easy to obtain all the remaining dots. It is straightforward to get the lucky sentence. Therefore, it is a bijection.