(5) Problem 5

(a) Since V is finite dimensional, $\dim V^{\otimes k} = (\dim V)^k = d^k$ (by Problem 1), so that

$$\operatorname{Hilb}(T(V);q) = \sum_{k=0}^{\infty} \dim\left(V^{\otimes k}\right) q^k = \sum_{k=0}^{\infty} d^k q^k$$

(b) Let $A = \langle u \otimes v - v \otimes u : \ u, v \in V \rangle$ be a homogenous ideal in $T(V)$, and let $A_k = A \cap V^{\otimes k}$. For every $k \geq 2$, A_k is all finite sums of k-tensors that look like this:

$$\begin{aligned} & a_1 \otimes \cdots \otimes (a_n \otimes a_{n+1} - a_{n+1} \otimes a_n) \otimes \cdots \otimes a_k \\ = \quad & (a_1 \otimes \cdots \otimes a_n \otimes a_{n+1} \otimes \cdots \otimes a_k) - (a_1 \otimes \cdots \otimes a_{n+1} \otimes a_n \otimes \cdots \otimes a_k). \end{aligned}$$

The quotient $V^{\otimes k}/A_k$ then has the property so that if you switch any two adjacent factors in a k-tensor, you have the same k-tensor. And since we can do this switching as many times as we need, all $k!$ permutations of the k factors in each k-tensors are equivalent. This makes the $\otimes$ operator symmetric, as suggested by the name "symmetric algebra" To get the Hilbert Series, note that $T(V)/A = \left(\bigoplus_{k=1}^{\infty} V^{\otimes k}\right) / \left(\bigoplus_{k=1}^{\infty} A_k\right) \cong \bigoplus_{k=1}^{\infty} \left(V^{\otimes k}/A_k\right)$, which is possible since A is homogenous in $T(V)$ (and $A = \bigoplus A_k$), so we see that we need to find $\dim V^{\otimes k}/A_k$. Let $\{v_1, \ldots, v_d\}$ be a basis for V assuming that $d \in \mathbb{N}$ is non-zero (otherwise this problem is trivial), so that

$$\{v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_k} : i_1, \ldots, i_k \in \{1, \ldots, d\}\}$$

is a basis for $V^{\otimes k}$. To get a basis for $V^{\otimes k}/A_k$ we ignore the order of the factors and only consider how many v_1's, how many v_2's, etc. are in each k-tuple. The basis in this case is

$$\left\{ \underbrace{v_1 \otimes \cdots}_{n_1 \text{times}} \otimes \cdots \otimes \underbrace{v_d \otimes \cdots}_{n_d \text{times}} : \, 0 \leq n_i \leq k, \, n_1 + \cdots + n_d = k \right\}$$

and counting this set is the same as counting the number of ways to draw $d-1$ vertical lines between k dots arranged horizontally to separate it into d different groups. For example, here we have $k = 11$ dots and 6 lines ($d = 7$):

$$|\,|\underbrace{\bullet \quad \bullet}_{v_3}|\underbrace{\bullet \quad \bullet \quad \bullet \quad \bullet}_{v_4}|\,|\underbrace{\bullet \quad \bullet \quad \bullet}_{v_6}|\underbrace{\bullet \quad \bullet}_{v_7}.$$

The corresponding basis element would be

$$v_3 \otimes v_3 \otimes v_4 \otimes v_4 \otimes v_4 \otimes v_4 \otimes v_6 \otimes v_6 \otimes v_6 \otimes v_7 \otimes v_7$$

In turn this line-dot scheme is the same as counting the number of ways to choose $d-1$ objects from $d+k-1$ objects. Thus,

$$\operatorname{Hilb}(S(V); q) = \sum_{k=0}^{\infty} \dim \left(V^{\otimes k}/A_k\right) q^k = \sum_{k=0}^{\infty} \binom{d+k-1}{d-1} q^k$$

In the special cases where $k = 0, 1$ this formula still holds because $A_0 = A_1 = \{0\}$, so $\dim V^{\otimes 0}/A_0 = \dim \mathbb{F}/\{0\} = \dim \mathbb{F} = 1 = \binom{d+0-1}{d-1}$ and $\dim V^{\otimes 1}/A_1 = \dim V = d = \binom{d}{d-1}$.

(c) Let $B = \langle v \otimes v \rangle$ be and ideal so that $\wedge(V) = T(V)/B$, and let $B_k = B \cap V^{\otimes k}$. The reasoning here will be in many ways identical to that in part (b) of this problem. We need to find $\dim V^{\otimes k}/B_k$, so we first claim that B_k is the set of all finite sums of the elements in

$$\{m_1 \otimes \cdots \otimes m_k : \, \exists i \neq j, \, m_i = m_j\}. \tag{0.1}$$

First, since B is generated by $v \otimes v$ and thus B_k has pure k-tensors where two adjacent factors are equal, then we know that the set given in 0.1 above is big enough. To show inclusion in the other direction, first note that for $v, v' \in V$

$$(v \otimes v') - (v' \otimes v) = (v' + v) \otimes (v' + v) - (v' \otimes v') - (v \otimes v) = 0$$

so that $\otimes$ is anti-symmetric. Thus, if a k-tensor has any two factors that are equal, through a series of switching adjacent factors and sign-changes, we can move those two factors so they are side-by-side, so the k-tensor is in B_k, showing that this set 0.1 is just the right size.

So, to find a basis for $V^{\otimes k}/B_k$ we just need to make sure that no two factors of a k-tensor are equal, when $k \geq 2$. That is if $\{v_1, \ldots, v_d\}$ is a basis for V, a basis for $V^{\otimes k}/B_k$ is

$$\{v_{n_1} \otimes \cdots \otimes v_{n_k} : \, 1 \leq n_1 < n_2 < \cdots < n_k \leq d\}.$$

The number of ways to do this is the same as the number of ways to choose k objects from d objects, or $\binom{d}{k}$. So we have

$$\operatorname{Hilb}(\wedge(V); q) = \sum_{k=0}^{\infty} \dim \left(V^{\otimes k}/B_k\right) q^k = \sum_{k=1}^{\infty} \binom{d}{k} q^k$$

and again this works for the special cases $k = 0, 1$, for which $V^{\otimes 0}/B_0 = \mathbb{F}$ and $V^{\otimes 1}/B_0 = V$, respectively.