4. *A Non-Commutative, Non-Cocommutative Bialgebra:* Let $q \in \mathbb{F}$ be non-zero. Consider the $\mathbb{F}$-algebra H_4 generated by indeterminates g and x subject to the relations $g^2 = 1, x^2 = 0$, and $xg = -gx$.

   (a) Show that $1, g, x$, and gx form a basis for H_4.

   Proof. We begin by showing that $H_4 \in \text{span}\{1, g, x, gx\}$. Note that $\text{span}\{gx\} = \text{span}\{xg\}$ because in an algebra we can freely scale by elements of the field, for example -1. Thus we can switch between xg and gx without altering the span of the set. This allows us to transform any expression composed of x and g into the form $g^i x^j$. Note that with the relations $x^2 = 0$ and $g^2 = 1$, we have $x^j = 0$ for any $j > 1$, and $g^i = g$ only for odd i, 1 otherwise. Thus the only possible unique (relative to the field) terms we can build from x and g are $1, g, x$, and gx. Thus $H_4 \in \text{span}\{1, g, x, gx\}$.

   To show linear dependence in these terms, assume we have some relation $\lambda_1 1 + \lambda_g g + \lambda_x x + \lambda_{gx} gx = 0$ where not all λ equal zero. If this were the case, such a relation would have to be given in the construction of the algebra. Since no such relation is given, we must have all $\lambda = 0$, which establishes linear independence. Thus, $1, g, x, gx$ form a basis for H_4. $\square$

   (b) Express the product $(a + bg + cx + dgx)(a' + b'g + c'x + d'gx)$ in terms of this basis, where $a, b, c, d, a', b', c', d' \in \mathbb{F}$.

   Proof.

$$(a + bg + cx + dgx)(a' + b'g + c'x + d'gx) =$$
$$aa' + ab'g + ac'x + ad'gx + ba'g + bb'g^2 + bc'gx + bd'g^2x + ...$$
$$ca'x + cb'xg + cc'x^2 + cd'xgx + da'gx + db'gxg + dc'gx^2 + dd'gxgx$$
$$= aa' + ab'g + ac'x + ad'gx + ba'g + bb' + bc'gx + bd'x + ...$$
$$ca'x - cb'gx + 0 + 0 + da'gx - db'x + 0 + 0$$
$$= (aa' + bb') + (ab' + ba')g + (ac' + bd' + ca' - db')x + (ad' + bc' - cb' + da')gx$$

   $\square$