3. Grouplike Elements in the Group Ring

In a coalgebra, we say the element x is *grouplike* if $\Delta(x) = x \otimes x$. Prove that in the group ring $\mathbb{F}[G]$, x is grouplike if and only if $x \in G$.

Proof. In the group ring $\mathbb{F}(G)$, we have comultiplication defined by $\Delta(g) = g \otimes g$ for $g \in G$, and extended linearly. Thus if $x \in G$, we have $\Delta(x) = x \otimes x$ by definition. What remains to show is that $\Delta(x) = x \otimes x \Rightarrow x \in G$.

Let $x \in \mathbb{F}(G)$, then $x = \sum_{g \in G} \lambda_g g$. Note that addition in the group ring is defined by addition in the field treating the group elements as the basis of a vector space. By the linearity of Δ, we have

$$\begin{aligned} \Delta x &= \Delta\left(\sum_{g \in G} \lambda_g g\right) \\ &= \sum_{g \in G} \lambda_g \Delta g \\ &= \sum_{g \in G} \lambda_g g \otimes g, \end{aligned}$$

and from our tensor axioms,

$$\begin{aligned} x \otimes x &= \left(\sum_{g \in G} \lambda_g g\right) \otimes \left(\sum_{h \in G} \lambda_h h\right) \\ &= \sum_{g,h \in G} \lambda_g \lambda_h g \otimes h. \end{aligned}$$

Equating these two, we get

$$\sum_{g \in G} \lambda_g g \otimes g = \sum_{g,h \in G} \lambda_g \lambda_h g \otimes h.$$

Since $g \otimes h$ does not appear on the left hand side of the equality, we must have $\lambda_g \lambda_h = 0$ for all $g \neq h$. Then if any $\lambda_g \neq 0$, we must have $\lambda_h = 0$ for all $h \neq g$. Then $x = \lambda_g g$ for some g. We now have

$$\lambda_g g \otimes g = \lambda_g^2 g \otimes g,$$

so we get $\lambda_g^2 = \lambda_g$. Then we get $\lambda_g = 0$ or $\lambda_g = 1$, so either $x = g$ or $x = 0$, where 0 is the additive identity in the group ring.

$\square$